Differentiable functions of Cayley-Dickson numbers.

S.V. Ludkovsky.

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Abstract

We investigate superdifferentiability of functions defined on regions of the real octonion (Cayley) algebra and obtain a noncommutative version of the Cauchy-Riemann conditions. Then we study the noncommutative analog of the Cauchy integral as well as criteria for functions of an octonion variable to be analytic. In particular, the octonion exponential and logarithmic functions are being considered. Moreover, superdifferentiable functions of variables belonging to Cayley-Dickson algebras (containing the octonion algebra as the proper subalgebra) finite and infinite dimensional are investigated. Among main results there are the Cayley-Dickson algebras analogs of Caychy's theorem, Hurtwitz', argument principle, Mittag-Leffler's, Rouche's and Weierstrass' theorems.

1 Introduction

Functions of real variables with values in Clifford algebras were investigated, for example, in [5]. In this article we continue our investigations of functions of variables belonging to noncommutative superalgebras [27] considering here functions of octonion variables and also of more general Cayley-Dickson algebras $\mathcal{A}_r$, $r \geq 4$, containing the octonion algebra as the proper subalgebra. The algebra of octonions is alternative, that gives possibility to define residues of functions in a reasonable way. The Cayley-Dickson algebras of larger dimensions are not already alternative and proceedings for them are

heavier. Nevertheless, using their power-associativity and distributivity it is possible to define differentiability of functions of variables belonging to the Cayley-Dickson algebra such that the differential has sufficiently well properties, that to define subsequently the line integral over such variable. This integral is extended on spaces of continuous functions along rectifiable paths. It is necessary to note, that the graded structure of the Cayley-Dickson algebra $\mathcal{A}_r$ over $\mathbf{R}$ and its noncommutativity for the real dimension not less, than 4 causes well properites of superdifferentiable functions $f : U \to \mathcal{A}_r$, where U is open in $\mathcal{A}_r^n$. Apart from the complex case a derivative f' of $f \in C^1(U, \mathcal{A}_r)$ is an operator and not in general a number even for $U \subset \mathcal{A}_r$, $n = 1$.

The theory of $\mathcal{A}_r$-holomorphic functions given below can not be reduced to the theory of holomorphic functions of several complex variables. Moreover, $\mathcal{A}_r$-holomorphic functions have many specific features in comparison with real locally analytic functions as, for example, the argument principle, homotopy theorem 2.15, theorem about representations of multiples of functions with the help of line integral along a loop, etc. show.

Dirac had used biquaternions (complexified quaternions) in his investigations of quantum mechanics. The Dirac equations $D_z f_1 - D_{\tilde{z}} f_2 = m(f_1 + f_2)$ and $D_{\tilde{z}} f_1 + D_z f_2 = m(f_1 - f_2)$ on the space of right superlinearly $(z, \tilde{z})$-superdifferentiable functions of quaternion variable can be extended on the space of $(z, \tilde{z})$-superdifferentiable functions $f_1(z, \tilde{z})$ and $f_2(z, \tilde{z})$ (see our definition in §2.2 below and in [27]) gives evident physical interpretation of a solution (f_1, f_2), $r \geq 2$, as spinors, where m is a mass of an elementary particle. We extend the operator $(D_z^2 + D_{\tilde{z}}^2)$ from the space of right superlinearly $(z, \tilde{z})$-superdifferentiable functions on the space of $(z, \tilde{z})$-superdifferentiable functions f, hence we get the Klein-Gordon equation $(D_z^2 + D_{\tilde{z}}^2) f = 2m^2 f$ in the particular case of quaternions and $\mathcal{A}_r$-algebras, $r \geq 3$. It is necessary to note that previous authors have used right (or left) superlinearly superdifferentiable functions, that does not form an algebra of functions and they have used multiple and iterated integrals and the Gauss-Ostrogradskii-Green formula, but they have not used line integrals over $\mathcal{A}_r$ (see, for example, [11] and references therein). While development their theory Yang and Mills known in theoretical and mathematical physics had actively worked with quaternions, but they have felt lack of the available theory of quaternion functions existing in their time. Yang also have expressed the idea, that possibly in quantum field theory it is worthwhile to use quaternion time (see page 198 [11]). It is known also the use of complex time through the Wick rotation in

quantum mechanics for getting solutions of problems, where the imaginary time is used for interpretations of probabilities of tunneling under energy barriers (walls). Using the special unitary group embedded into the quaternion skew field $\mathbf{H}$ it makes equivalent under isomoprphism with $SO(3)$ all spatial axes. On the other hand, the major instrument for measurement is the spectrum. When there are deep energy wells or high energy walls, then it makes obstacles for penetrating electromagnetic waves and radiation, that is well known also in astronomy, where black holes are actively studied (see page 199 and §3.b [11] and references therein). W. Hamilton in his lectures on quaternions also tackled a question of events in $\mathbf{H}$ and had thought about use of quaternions in astronomy and celestial mechanics (see [12, 30] and references therein). Therefore, in general to compare the sequence of events it may be necessary in definite situations to have the same dimensional time space as the coordinate space. On the other hand, spatial isotropy at least local in definite domains makes from each axis under rotations and dilatations $SU(2) \times \mathbf{R}$ isomorphic with $\mathbf{H}$. Therefeore, it appears that in definite situations it would be sufficent to use $\mathbf{H}^4$ instead of the Minkowski space-time $\mathbf{R}^{1,3}$, where $\mathbf{R}^{1,3}$ has the embedding into $\mathbf{H}$. Since $\mathbf{H}$ as the $\mathbf{R}$-linear space is isomorphic with $\mathbf{R^4}$, then there exists the embedding ζ of $\mathbf{R^{1,3}}$ into $\mathbf{H}$ such that $\zeta(x_1, x_i, x_j, x_k) = x_1 + x_i i + x_j j + x_k k$, where the $\mathbf{R^{1,3}}$-norm is given by the equation $|x|_{1,3} = (x^2 + \tilde{x}^2)/2 = Re(x^2) = x_1^2 - x_i^2 - x_j^2 - x_k^2$ and the $\mathbf{R^{1,3}}$ scalar product is given by the equality $(x, y)_{1,3} := (xy + \tilde{y}\tilde{x})/2 = Re(xy)$ $= x_1y_1 - x_iy_i - x_jy_j - x_ky_k$, where $x = x_1 + x_i i + x_j j + x_k k$, $x_1, ..., x_k \in \mathbf{R}$ (see §2.1). This also can be used for embeddings of hyperbolic manifolds into quaternion manifolds. Then $\mathbf{H}^4$ can be embedded into the algebra $\mathcal{A}_4$ of sedenions. It is also natural for describing systems with spin, isospin, flavor, color and their interactions. The enlargement of the space-time also is dictated in some situations by symmetry properties of differential equations or a set of operators describing a system. For example, special unitary groups $SU(n)$, for $n = 3, 5-8, 11$, etc., exceptional Lie groups, are actively used in theory of elementary particles [11], but these groups can be embedded into the corresponding Cayley-Dickson algebra $\mathcal{A}_r$ [1]. Indeed, $U(m) \subset GL(n, \mathbf{C}) \subset \mathbf{C}^{n^2}$ while $\mathbf{C}^m$ has the embedding into $\mathcal{A}_r$, where $m = 2^{r-1}$, such that $\mathbf{C}^m \ni (x^1 + iy^1, ..., x^m + iy^m) =: \xi \mapsto z := (x^1 + i_1y^1) + i_2(x^2 + i_2^* i_3 y^2) + ... + i_{2m-2}(x^m + i_{2m-2}^* i_{2m-1} y^m) \in \mathcal{A}_r$, since $(i_l^* i_k)^2 = -1$ for each $l \ne k \ge 1$, where $\{i_0, i_1, ..., i_{2m-1}\}$ is the basis of generators of $\mathcal{A}_r$, $i_0 = 1$, $i_k^2 = -1$, $i_0 i_k = i_k i_0$, $i_l i_k = -i_k i_l$ for each $k \ne l \ge 1$,

$i = (-1)^{1/2}$, $z^* = x^1 - i_1y^1 - i_2x^2 - i_3y^2 - ... - i_{2m-2}x^m - i_{2m-1}y^m$, the norm $(zz^*)^{1/2} =: |z| = (\sum_{k=1}^m |x^k + iy^k|^2)^{1/2} =: |\xi|$ satisfies the parallelogramm identity and induces the scalar product.

This paper as the previous one [27] is devoted to the solution of the W. Hamilton problem of developing line integral and holomorphic function theory of quaternion variables, but now we consider general case of Cayley-Dickson algebras variables.

In this paper we investigate differentiability of functions defined on regions of the real octonion (Cayley) algebra. For this we consider specific definition of superdifferentiability and obtain a noncommutative version of the Cauchy-Riemann conditions in the particular case of right superlinear superdifferentiability. Then we study the noncommutative analog of the Cauchy integral as well as criteria for functions of an octonion variable to be analytic. In particular, the octonion exponential and logarithmic functions are being considered. Moreover, superdifferentiable functions of variables belonging to Cayley-Dickson algebras (containing the octonion algebra as the proper subalgebra) finite and infinite dimensional are investigated. Among main results there are the Cayley-Dickson algebras analogs of Caychy's theorem, Hurtwitz', argument principle, Mittag-Leffler's, Rouche's and Weierstrass' theorems.

The results of this paper can serve for subsequent investigations of special functions of Cayley-Dickson algebra variables, noncommutative sheaf theory, manifolds of noncommutative geometry over Cayley-Dickson algebras, their groups of loops and diffeomorphisms (see also [23, 24, 25, 26, 29]).

2 Differentiability of functions of octonion variables

To avoid misunderstandings we first introduce notations.

2.1. We write $\mathbf{H}$ for the skewfield of quaternions over the real field $\mathbf{R}$ with the classical quaternion basis 1, i, j, k, satisfying relations of $\mathcal{A}_2$ (see Introduction). The quaternion skewfield $\mathbf{H}$ has an anti-automorphism η of order two $\eta : z \mapsto \tilde{z}$, where $\tilde{z} = w_1 - w_i i - w_j j - w_k k$, $z = w_1 + w_i i + w_j j + w_k k$; $w_1, ..., w_k \in \mathbf{R}$. There is a norm in $\mathbf{H}$ such that $|z| = |z\tilde{z}|^{1/2}$, hence $\tilde{z} = |z|^2 z^{-1}$.

The algebra $\mathbf{K}$ of octonions (octaves, the Cayley algebra) is defined as an eight-dimensional algebra over $\mathbf{R}$ with a basis, for example,

(1) $\mathbf{b}_3 := \mathbf{b} := \{1, i, j, k, l, il, jl, kl\}$ such that

(2) $i^2 = j^2 = k^2 = l^2 = -1$, $ij = k$, $ji = -k$, $jk = i$, $kj = -i$, $ki = j$, $ik = -j$, $li = -il$, $jl = -lj$, $kl = -lk$;

(3) $(\alpha + \beta l)(\gamma + \delta l) = (\alpha\gamma - \tilde{\delta}\beta) + (\delta\alpha + \beta\tilde{\gamma})l$

is the multiplication law in $\mathbf{K}$ for each α, β, γ, $\delta \in \mathbf{H}$, $\xi := \alpha + \beta l \in \mathbf{K}$, $\eta := \gamma + \delta l \in \mathbf{K}$, $\tilde{z} := v - wi - xj - yk$ for a quaternion $z = v + wi + xj + yk \in \mathbf{H}$ with $v, w, x, y \in \mathbf{R}$.

The octonion algebra is neither commutative, nor associative, since $(ij)l = kl$, $i(jl) = -kl$, but it is distributive and $\mathbf{R}1$ is its center. If $\xi := \alpha + \beta l \in \mathbf{K}$, then

(4) $\tilde{\xi} := \tilde{\alpha} - \beta l$ is called the adjoint element of ξ, where $\alpha, \beta \in \mathbf{H}$. Then

(5) $(\xi\eta)\tilde{} = \tilde{\eta}\tilde{\xi}$, $\tilde{\xi} + \tilde{\eta} = (\xi + \eta)\tilde{}$ and $\xi\tilde{\xi} = |\alpha|^2 + |\beta|^2$,

where $|\alpha|^2 = \alpha\tilde{\alpha}$ such that

(6) $\xi\tilde{\xi} =: |\xi|^2$ and $|\xi|$ is the norm in $\mathbf{K}$. Therefore,

(7) $|\xi\eta| = |\xi||\eta|$,

consequently, $\mathbf{K}$ does not contain divisors of zero (see also [17, 21, 35]). The multiplication of octonions satsifies equations:

(8) $(\xi\eta)\eta = \xi(\eta\eta)$,

(9) $\xi(\xi\eta) = (\xi\xi)\eta$,

that forms the alternative system. In particular, $(\xi\xi)\xi = \xi(\xi\xi)$. Put $\tilde{\xi} = 2a - \xi$, where $a = Re(\xi) := (\xi + \tilde{\xi})/2 \in \mathbf{R}$. Since $\mathbf{R}1$ is the center of $\mathbf{K}$ and $\tilde{\xi}\xi = \xi\tilde{\xi} = |\xi|^2$, then from $(8, 9)$ by induction it follows, that for each $\xi \in \mathbf{K}$ and each n-tuplet (product), $n \in \mathbf{N}$, $\xi(\xi(...\xi\xi)...) = (...(\xi\xi)\xi...)\xi$ the result does not depend on an order of brackets (order of consequtive multiplications), hence the definition of $\xi^n := \xi(\xi(...\xi\xi)...)$ does not depend on the order of brackets. This also shows that $\xi^m\xi^n = \xi^n\xi^m$, $\xi^m\tilde{\xi}^m = \tilde{\xi}^m\xi^n$ for each $n, m \in \mathbf{N}$ and $\xi \in \mathbf{K}$. Apart from the quaternions, the octonion algebra can not be realized as the subalgebra of the algebra $\mathbf{M}_8(\mathbf{R})$ of all 8×8-matrices over $\mathbf{R}$, since $\mathbf{K}$ is not associative, but $\mathbf{M}_8(\mathbf{R})$ is associative. The noncommutative nonassociative octonion algebra $\mathbf{K}$ is the $\mathbf{Z_2}$-graded $\mathbf{R}$-algebra $\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_1$, where elements of $\mathbf{K}_0$ are <u>*even*</u> and elements of $\mathbf{K}_1$ are <u>*odd*</u> (see, for example, [20, 21, 34]). There are the natural embeddings $\mathbf{C} \hookrightarrow \mathbf{K}$ and $\mathbf{H} \hookrightarrow \mathbf{K}$, but neither $\mathbf{K}$ over $\mathbf{C}$, nor $\mathbf{K}$ over $\mathbf{H}$, nor $\mathbf{H}$ over $\mathbf{C}$ are algebras, since the centres of them are $Z(\mathbf{H}) = Z(\mathbf{K}) = \mathbf{R}$.

We consider also the Cayley-Dickson algebras $\mathcal{A}_n$ over $\mathbf{R}$, where 2^n is its

dimension over $\mathbf{R}$. They are constructed by induction starting from $\mathbf{R}$ such that $\mathcal{A}_{n+1}$ is obtained from $\mathcal{A}_n$ with the help of the doubling procedure, in particular, $\mathcal{A}_0 := \mathbf{R}$, $\mathcal{A}_1 = \mathbf{C}$, $\mathcal{A}_2 = \mathbf{H}$, $\mathcal{A}_3 = \mathbf{K}$ and $\mathcal{A}_4$ is known as the sedenion algebra [1]. The Cayley-Dickson algebras are $*$-algebras, that is, there is a real-linear mapping $\mathcal{A}_n \ni a \mapsto a^* \in \mathcal{A}_n$ such that

(10) $a^{**} = a$,

(11) $(ab)^* = b^*a^*$ for each $a, b \in \mathcal{A}_n$. Then they are nicely normed, that is,

(12) $a + a^* =: 2Re(a) \in \mathbf{R}$ and

(13) $aa^* = a^*a > 0$ for each $0 \ne a \in \mathcal{A}_n$. The norm in it is defined by

(14) $|a|^2 := aa^*$. We also denote a^* by $\tilde{a}$. Each $0 \ne a \in \mathcal{A}_n$ has a multiplicative inverse given by $a^{-1} = a^*/|a|^2$.

The doubling procedure is as follows. Each $z \in \mathcal{A}_{n+1}$ is written in the form $z = a + bl$, where $l^2 = -1$, $l \notin \mathcal{A}_n$, $a, b \in \mathcal{A}_n$. The addition is componentwise. The conjugate is

(15) $z^* := a^* - bl$.

The multiplication is given by Equation (3).

The basis of $\mathcal{A}_n$ over $\mathbf{R}$ is denoted by $\mathbf{b}_n := \mathbf{b} := \{1, i_1, ..., i_{2^n-1}\}$, where $i_s^2 = -1$ for each $1 \le s \le 2^n - 1$, $i_{2^{n-1}} := l$ is the additional element of the doubling procedure of $\mathcal{A}_n$ from $\mathcal{A}_{n-1}$, choose $i_{2^{n-1}+m} = i_m l$ for each $m = 1, ..., 2^{n-1} - 1$, $i_0 := 1$.

An algebra is called alternative, if each its subalgebra generated by two elements is associative. An algebra is called power-associative, if its any subalgebra generated by one element is associative. Only for $n = 0, ..., 3$ the Cayley-Dickson algebras $\mathcal{A}_n$ are division alternative algebras. For $n \ge 4$ the Cayley-Dickson algebras $\mathcal{A}_n$ are not division algebras, but they are power-associative. To verify the latter property consider $z \in \mathcal{A}_n$ written in the form $z = v + M$, where $v = Re(z)$, $M := (z - z^*)/2 =: Im(z)$. Then v and M commute and they are orthogonal, $M^* = -M$. Therefore, the subalgebra generated by z is associative if and only if the algebra generated by M is associative. Since $M^*M = MM^* = |M|^2$ and $M^* = -M$, then the subalgebra generated by M is associative.

Verify that z and $\tilde{z}$ in $\mathcal{A}_r$ are independent variables. Suppose contrary that there exists $\gamma \in \mathcal{A}_r$ such that $z + \gamma\tilde{z} = 0$ for each $z \in \mathcal{A}_r$. Write $z = a+bl$, $\gamma = \alpha+\beta l$, where $a, b, \alpha, \beta \in \mathcal{A}_{r-1}$. Then $z+\gamma\tilde{z} = 0$ is equivalent to $\alpha a^* + b^*\beta = -a$ and $-b\alpha + \beta a = -b$. Consider z with $a \ne 0$ and $b \ne 0$. Then from the latter two equations we get: $\alpha = -(a + b^*\beta)a|a|^{-2} = 1 + b^*\beta a|b|^{-2}$,

since $a^{-1} = a^*|a|^{-2}$. This gives $\beta = -2bRe(a)|z|^{-2}$ and $\alpha = 1-2aRe(a)|z|^{-2}$. Taking in particular $|z| = 1$ and $Re(a) \ne 0$ and varying z we come to the contardiction, since γ is not a constant. Therefore, z and z^* are two variables in $\mathcal{A}_r$ left-linearly (or right-linearly) independent over $\mathcal{A}_r$.

For each $\mathcal{A}_r$ with $r \ge 2$, $r \in \mathbf{N}$, there are the identities: $z + z^* = 2w_1$, $s(zs^*) = z^* + 2w_s s$ for each $s \in \hat{b}$, where $z = \sum_{s\in\mathbf{b}} w_s s$, $w_s \in \mathbf{R}$ for each $s \in \mathbf{b} := \{1, i_1, ..., i_{2^r-1}\}$, $\hat{b} := \mathbf{b}\setminus\{1\}$, hence $z^* = (2^r-2)^{-1}\{-z+\sum_{s\in\hat{b}} s(zs^*)\}$ for each $r \ge 2$, $r \in \mathbf{N}$. Therefore, z^* does not play so special role in $\mathcal{A}_r$, $r \ge 2$, as for $\mathbf{C}$.

In view of noncommutativity of $\mathcal{A}_r$, $r \ge 3$, and the identities in it caused by the conjugation (15), multiplication (3) and addition laws, for example, Equations $(7, 8, 9)$ for octonions, also Equations $(10, 11, 14)$ and Conditions $(12, 13)$ in the general case of $\mathcal{A}_r$ a polynomial function $P : U \to \mathcal{A}_r$ in variables z and z^{-1} may have several different representations

$$\check{\mathrm{P}}(z) = \sum_{k,q(m)} \{b_{k,1}z^{k_1}...b_{k,m}z^{k_m}\}_{q(m)},$$

where $b_{k,j} \in \mathcal{A}_r$ are constants, $k = (k_1, ..., k_m)$, $m \in \mathbf{N}$, $k_j = (k_{j,1}, ..., k_{j,n})$, $k_{j,l} \in \mathbf{Z}$, $z^{k_j} := {}^1z^{k_{j,1}}... {}^nz^{k_{j,n}}$, ${}^lz^0 := 1$, U is an open subset of $\mathcal{A}_r^n$. Certainly, we can consider $z - z_0$ instead of z in the formula of $\check{\mathrm{P}}(z)$ on the right side, when a marked point z_0 is given. In view of the nonassociativity of $\mathcal{A}_r$ here is used the notation $\{a_1...a_m\}_{q(m)}$ of the product of elements $a_1, ..., a_m \in \mathcal{A}_r$ corresponding to the order of products in this term defined by the position of brackets $q(m) := (q_m, ..., q_3)$, where $a_v := (b_{k,v}z^{k_v})$ for each $v = 1, ..., m$, $q_m \in \mathbf{N}$ means that the first (the most internal bracket) corresponds to the multiplication $a_{q_m}a_{q_m+1}$ such that to the situation $(a_1...(a_ta_{t+1})...(a_wa_{w+1})...a_m)$ with formally two simultaneous independent multiplications, but $t < w$ by our definition of ordering there corresponds $q_m = t$. After the first multiplication we get the product of ${a'}_1, ..., {a'}_{m-1} \in \mathcal{A}_r$, where not less, than $m-2$ of these elements are the same as in the preceding term, then q_{m-1} corresponds to the first multiplication in this new term. We omit q_2 and q_1, since they are unique. Each term $\{b_{k,1}z^{k_1}...b_{k,m}z^{k_m}\}_{q(m)} =: \omega(b_k, z) \ne 0$ we consider as a word of length $\xi(\omega) = \sum_{j,l} \delta(k_{j,l}) + \sum_j \kappa(b_{k,j})$, where $\delta(k_{j,l}) = 0$ for $k_{j,l} = 0$ and $\delta(k_{j,l}) = 1$ for $k_{j,l} \ne 0$, $\kappa(b_{k,j}) = j$ for $b_{k,j} = 1$, $\kappa(b_{k,j}) = j + 1$ for $b_{k,j} \in \mathcal{A}_r \setminus \{0, 1\}$. A polynomial P is considered as a phrase $\check{\mathrm{P}}$ of a length $\xi(\check{\mathrm{P}}) := \sum_k \xi(\omega(b_k, z))$. Using multiplication of constants in $\mathcal{A}_r$, commutativity of $v \in \mathbf{R}$ with each ${}^lz$ and ${}^l\tilde{z}$, and ${}^lz^a\,{}^lz^b = {}^lz^{a+b}$ and ${}^l\tilde{z}^a\,{}^l\tilde{z}^b = {}^l\tilde{z}^{a+b}$,

${}^l z\,{}^l\tilde{z} = {}^l\tilde{z}\,{}^l z$, it is possible to consider representations of P as phrases $\check{\mathrm{P}}$ of a minimal lenght $\xi(\check{\mathrm{P}})$, then order them lexicographically by vectors $q(m)$. We choose one such $\check{\mathrm{P}}$ of a minimal lenght and then minimal with respect to the lexicographic ordering of $q(m)$. In view of the commutativity of the addition for terms $\{a_1...a_m\}_{q(m)}$ and $\{a'_1...a'_m\}_{q'(m)}$ of equal length and having different vectors $q(m)$ and $q'(m)$ the order of $q(m)$ and $q'(m)$ for $\check{\mathrm{P}}$ is not important such that the minimality is tested by all orderings of $q(m)$ among all terms of a given lenght in $\check{\mathrm{P}}$.

If $f : U \to \mathcal{A}_r$ is a function presented by a convergent by z series $f(z) = \sum_n P_n(z)$, where $P_n(vz) = v^n P_n(z)$ for each $v \in \mathbf{R}$ is a $\mathbf{R}$-homogeneous polynomial, $n \in \mathbf{Z}$, then we consider among all representations of f such for which $\xi(\check{\mathrm{P}}_n)$ is minimal for each $n \in \mathbf{Z}$. This serves us to find representatives in classes of equivalent elements of the $\mathbf{R}$-algebra of all polynomials on U and z-analytic functions. The corresponding family of locally z-analytic functions on U is denoted by $C^\omega_z(U, \mathcal{A}_r)$ or $C^\omega(U, \mathcal{A}_r)$. Each element of $C^\omega_z(U, \mathcal{A}_r)$ by our definition is a unique phrase which may be infinite. We do not exclude a possibility that two different phrases f and g may have the same set-theoretical graph $\Gamma_f := \{(z, f(z)) : z \in U\}$ as mappings from U into $\mathcal{A}_r$, for example, when a class of equivalence defined by a graph has nonunique element of minimal length. If each P_n for f has a decomposition of a particular left type

$$\check{\mathrm{P}}(z) = \sum_k b_k(z^k),$$

where $0 \le k \in \mathbf{Z}$, $b_k \in \mathcal{A}_r$, then the space of all such locally analytic functions on U is denoted by ${}_lC^\omega(U, \mathcal{A}_r)$. The space of locally analytic functions f having right type decompositions for each P_n

$$\check{\mathrm{P}}(z) = \sum_k (z^k) b_k$$

is denoted by ${}_rC^\omega(U, \mathcal{A}_r)$. The corresponding space in variables $(z, \tilde{z})$ is denoted by $C^\omega_{z,\tilde{z}}(U, \mathcal{A}_r)$ and in variables $\tilde{z}$ by $C^\omega_{\tilde{z}}(U, \mathcal{A}_r)$, where $C^\omega_{z,\tilde{z}} := C^\omega_{{}_1z,\ {}_2z}(U^2, \mathcal{A}_r)|_{{}_1z=z,\ {}_2z=\tilde{z}}$, ${}_1z$ and ${}_2z \in U$. By our definition each element of $C^\omega_{z,\tilde{z}}(U, \mathcal{A}_r)$ is a unique phrase which may be infinite.

The $\mathbf{R}$-linear space $C^\omega_{z,\tilde{z}}(U, \mathcal{A}_r)$ is dense in the $\mathbf{R}$-linear space $C^0(U, \mathcal{A}_r)$ of all continuous functions $f : U \to \mathcal{A}_r$. We denote by $C^0_z(U, \mathcal{A}_r)$ the $\mathbf{R}$-linear space of all equivalence classes of Cauchy sequences from $C^\omega_z(U, \mathcal{A}_r)$

converging relative to the C^0-uniformity. Analogously we define $C^0_{\tilde{z}}(U,\mathcal{A}_r)$ and $C^0_{z,\tilde{z}}(U,\mathcal{A}_r)$.

2.1.1. Definition. If $\mathcal{G}\in C^0_z(U,\mathcal{A}_r)$, then we say that $\mathcal{G}$ is z-represented. Elements of $C^0_{\tilde{z}}$ (or $C^0_{z,\tilde{z}}$) we call $\tilde{z}$- (or $(z,\tilde{z})-$ respectively) represented functions. If $f:U\to\mathcal{A}_r$ is a (set-theoretic) continuous function and $\mathcal{G}\subset\mathcal{F}\in C^0_{z,\tilde{z}}(U,\mathcal{A}_r)$ such that each Cauchy sequence $\{\zeta_n : n\in\mathbf{N}\}$ from a family $\mathcal{G}$ (of converging Cauchy sequences) converges to f relative to the C^0-uniformity, then we call each $g\in\mathcal{G}=\mathcal{G}(f)$ an algebraic continuous function g. (Since $\mathcal{F}$ is an equivalence class, then each $\{\zeta_n : n\}\in\mathcal{G}$ converges to the same limit f).

We say that $\mathcal{G}$ posses property A, if each $\{\zeta_n : n\}\in\mathcal{G}$ posseses property A. We say that f posseses property A, if there exists $\mathcal{G}(f)$ possesing property A and such that $\mathcal{G}(f)=\mathcal{H}$, where either $\mathcal{H}\in C^0_z(U,\mathcal{A}_r)$ or $\mathcal{H}\in C^0_{\tilde{z}}(U,\mathcal{A}_r)$, or $\mathcal{H}\in C^0_{z,\tilde{z}}(U,\mathcal{A}_r)$. If $\mathcal{G}\ne\emptyset$, $\mathcal{G}\subset\mathcal{H}$ and $\mathcal{G}\ne\mathcal{H}$ we will talk about $(A,\mathcal{G})$ property. If f is of higher class of smoothness C^n, C^∞ or C^ω, etc., then we take intersections of $\mathcal{G}(f)$ and $\mathcal{H}$ with C^n or C^∞, or C^ω, etc. supposing convergence relative to the respective uniformity and such that $\mathcal{G}(f)\cap C^n$ or $\mathcal{G}(f)\cap C^\infty$, or $\mathcal{G}(f)\cap C^\omega$, etc. is nonvoid. Writing arguments $({}_1z,...,{}_nz)$ of f we will outline the situation in each case indicating a subset of variables by which property A is accomplished. We may write for short $f(z)$ or f instead of $f(z,\tilde{z})$ in situations, in which it can not cause a confusion. Our general supposition is that a continuous function has a $(z,\tilde{z})$-representation, if another will not be specified.

2.1.2. Proposition. *Let A be an $\mathcal{A}_r$-additive, $\mathbf{R}$-homogeneous operator $A:\mathcal{A}_r^n\to\mathcal{A}_r^n$, $r\ge 2$, $n\ge 1$, then A is $\mathbf{R}$-linear and there exists a finite family A_j of right-$\mathcal{A}_r$-linear and B_j of left-$\mathcal{A}_r$-linear operators $j\in\{1,2,...,2^r\}$ independent of $h\in\mathcal{A}_r^n$ such that $A(h)=\sum_j A_j(hB_j)$ for each $h\in\mathcal{A}_r^n$, where we write $A_j(h)=: A_jh$ and $B_j(h)=: hB_j$.*

Proof. The first statement is evident. To prove the second mention that $A(h)=\sum_{j=0}^{2^r-1}A(i_j)h_j$, where $h_j\in\mathbf{R^{n2^r}}$, $h=\sum_{j=0}^{2^r-1}h_ji_j$, $A(i_j)$ are $\mathbf{R}$-linear operators independent of h_j. On the other hand,
$h_j=(-hi_j+i_j(2^r-2)^{-1}\{-h+\sum_{j=1}^{2^r-1}i_j(hi_j^*)\})/2$ for each $j=1,2,...,2^r-1$,
$h_0=(h+(2^r-2)^{-1}\{-h+\sum_{j=1}^{2^r-1}i_j(hi_j^*)\})/2$.
Substituting these expressions of h_j, $j=0,1,...,2^r-1$, into each term $A(i_j)h_j$ we get the second statement.

2.2. Definition. Consider an open region U in $\mathcal{A}_r^n$, $r\ge 3$, the n-fold

product of copies of $\mathcal{A}_r$, and let $f : U \to \mathcal{A}_r$ be a function. Then f is said to be z-superdifferentiable at a point $({}^1z, ..., {}^nz) = z \in U$, ${}^1z, ..., {}^nz \in \mathcal{A}_r$, if it satisfies Conditions $(2-7)$ below and if it can be written in the form

$$(1) \quad f(z+h) = f(z) + \sum_{j=1}^{n} A_j \, {}^jh + \epsilon(h)|h|$$

for each $h \in \mathcal{A}_r^n$ such that $z+h \in U$, where A_j is an $\mathcal{A}_r$-valued $\mathcal{A}_r$-additive $\mathbf{R}$-homogeneous operator of h-variable, in general it is non-linear for each $j = 1, ..., n$ and A_j is denoted by $(Df(z)).e_j$ and there exists a derivative $f'(z)$ such that a differential is given by

$$(2) \quad Df(z).h := f'(z).h := \sum_{j=1}^{n} (\partial f(z)/\partial \, {}^jz) \, {}^jh,$$

where $\epsilon(h)$, $\epsilon : \mathcal{A}_r^n \to \mathcal{A}_r$, is a function continuous at zero such that $\epsilon(0) = 0$, $e_j = (0, ..., 0, 1, 0, ..., 0)$ is the vector in $\mathcal{A}_r^n$ with 1 on j-th place,

$$(3) \quad Df(z).h =: (Df)(z; h)$$

such that $(Df)(z;h)$ is additive in h and $\mathbf{R}$-homogeneous, that is,

$$(4) \quad (Df)(z; h_1 + h_2) = (Df)(h_1) + (Df)(h_2) \text{ and } (Df)(z; vh) = v(Df)(z; h)$$

for each h_1, h_2 and $h \in \mathcal{A}_r^n$, $v \in \mathbf{R}$. There are imposed conditions:

$$(5) \quad (\partial_z z).h = h, \quad \partial_z 1 = 0, \quad \partial_z \tilde{z} = 0, \quad \partial_{\tilde{z}} z = 0, \quad (\partial_{\tilde{z}} \tilde{z}).h = \tilde{h}$$

$$\text{also } D = \partial_z + \partial_{\tilde{z}}, \quad (D(fg)).h = ((Df).h)g + f(Dg).h$$

for a product of two supedifferentiable functions f and g and each $h \in \mathcal{A}_r^n$, where the notation ∂_z corresponds to $\partial/\partial z$ and $\partial_{\tilde{z}}$ corresponds to $\partial/\partial \tilde{z}$. We also have distributivity laws relative to multiplication from the right by elements $\lambda \in \mathcal{A}_r$:

$$(6) \quad (D(f+g))(z; h\lambda) = (Df)(z; h\lambda) + (Dg)(z; h\lambda),$$

$$(Df)(z; h(\lambda_1 + \lambda_2)) = (Df)(z; h\lambda_1) + (Df)(z; h\lambda_2)$$

for each superdifferentiable functions f and g at z and each λ, λ_1 and $\lambda_2 \in \mathcal{A}_r$. There are also left distributive laws:

$$(7) \quad (D\lambda(f+g))(z;h) = \lambda(Df)(z;h) + \lambda(Dg)(z;h),$$

$$(D(\lambda_1+\lambda_2)f)(z;h) = \lambda_1(Df)(z;h) + \lambda_2(Df)(z;h).$$

If use $(z,\tilde{z})$-representation of polynomials and functions, then we define $(z,\tilde{z})$-superdifferentiability by the pair $(z,\tilde{z})$, by z and by $\tilde{z}$ such that
(8) $D_z\tilde{z} = 0$, $D_{\tilde{z}}z = 0$, $(D_z z).h = h$, $(D_{\tilde{z}}\tilde{z}).h = \tilde{h}$,
$(D_{z,\tilde{z}}(fg)).h = ((D_{z,\tilde{z}}f).h)g + f(D_{z,\tilde{z}}g).h$ and $(D_{z,\tilde{z}}f).h = (D_z f).h + (D_{\tilde{z}}f).h$ for each two $(z,\tilde{z})$-superdifferentiable functions f and g, each $h \in \mathcal{A}_r^n$ (see also [27]). We take a function $g({}_1z, {}_2z)$ in the z-representation by ${}_1z$ and ${}_2z$, then consider the operator D by the variable $({}_1z, {}_2z)$ and in the expression $(Dg({}_1z, {}_2z)).h$, put for the components ${}_1z = z$, ${}_2z = \tilde{z}$, ${}_1h = {}_2h =: \alpha \in \mathcal{A}_r^n$ and consider the function $g(z,\tilde{z}) =: f$, where $z = ({}^1z, ..., {}^nz) \in U \subset \mathcal{A}_r^n$, $\tilde{z} = ({}^1\tilde{z}, ..., {}^n\tilde{z})$, $az := (a\,{}^1z, ..., a\,{}^nz)$, $zb := ({}^1zb, ..., {}^nzb)$ for each $a, b \in \mathcal{A}_r$.

If there is a function $g({}_1z, {}_2z)$ on an open subset W in $\mathcal{A}_r^{2n}$ with values in $\mathcal{A}_r$ $({}_1z, {}_2z)$-superdifferentiable at a point $({}_1z, {}_2z)$, ${}_1z$ and ${}_2z \in \mathcal{A}_r^n$, also
$g({}_1z, {}_2z)|_{{}_1z=z,\ {}_2z=\tilde{z}} =: f(z,\tilde{z})$, $z = \xi$,
then we say that f is $(z,\tilde{z})$-superdifferentiable at a point ξ and
(9) $(D_z f(z,\tilde{z})).h = (\partial f(z,\tilde{z})/\partial z).h := \{(Dg({}_1z, {}_2z)).(h,0)\}|_{{}_1z=z,\ {}_2z=\tilde{z}}$,
$(D_{\tilde{z}} f(z,\tilde{z})).h = (\partial f(z,\tilde{z})/\partial \tilde{z}).h := \{(Dg({}_1z, {}_2z)).(0,h)\}|_{{}_1z=z,\ {}_2z=\tilde{z}}$,
where $h \in \mathcal{A}_r^n$ and f is supposed to be defined by g and its restriction on $\{({}_1z, {}_2z) \in W : {}_2z = \tilde{({}_1z)}\}$,
(10) $D_{{}_1z}g({}_1z, {}_2z).h := D_{({}_1z,\ {}_2z)}g({}_1z, {}_2z).(h,0)$,
$D_{{}_2z}g({}_1z, {}_2z).h := D_{({}_1z,\ {}_2z)}g({}_1z, {}_2z).(0,h)$,
$(D_z f(z,\tilde{z})).e_j =: \partial f(z,\tilde{z})/\partial\,{}^jz$, $(D_{\tilde{z}} f(z,\tilde{z})).e_j =: \partial f(z,\tilde{z})/\partial\,{}^j\tilde{z}$.
Since the Cayley-Dickson algebras are over $\mathbf{R}$ and Fréchet differentials are unique, then for functions $g : W \to \mathcal{A}_r$ and $f : U \to \mathcal{A}_r$, their superdifferentials $(Dg).h$ and $(D_{(z,\tilde{z})}f).\alpha$ are unique, so we have $D_z = D_{{}_1z}|_{{}_1z=z}$, $D_{\tilde{z}} = D_{{}_2z=\tilde{z}}$ in the $(z,\tilde{z})$-representation, where U is open in $\mathcal{A}_r^n$ such that $\{({}_1z = z, {}_2z = \tilde{z}) : z \in U\} \subset W$. In particular, if there are functions f_1, f_2, f_3 such that $f_3 = f_1(z,\tilde{z})f_2(z,\tilde{z})$, $f_j = g_j({}_1z, {}_2z)|_{{}_1z=z,\ {}_2z=\tilde{z}}$, $j = 1, 2, 3$, where either g_j for each j is presented by a minimized series of §2.1, or while multiplication $(f_1, f_2) \mapsto f_1f_2$ no any reorganization of their series, for example, by minimality is made (this is the case of our definition above), then

(11) $(D_z f_1 f_2).h = ((D_z f_1).h)f_2 + f_1(D_z f_2).h$ and
$(D_{\tilde{z}} f_1 f_2).h = ((D_{\tilde{z}} f_1).h)f_2 + f_1(D_{\tilde{z}} f_2).h$
for each $h \in \mathcal{A}_r^n$, since $D_{z,\tilde{z}} = D_z + D_{\tilde{z}}$. Generally,
$(D_{z,\tilde{z}} f_1 f_2).h = ((D_{z,\tilde{z}} f_1).h)f_2 + f_1(D_{z,\tilde{z}} f_2).h$ for each $(z,\tilde{z})$-superdifferentiable functions f_1 and f_2 on U and each $h \in \mathcal{A}_r^n$.

A function $f : U \to \mathcal{A}_r$ is called $\tilde{z}$-superdifferentiable at a point ξ, if there exists a function $g : U \to \mathcal{A}_r$ such that $g(\tilde{z}) = f(z)$ and $g(z)$ is z-superdifferentiable at ξ.

Notation. We may write a function $f(z)$ with $z \in \mathcal{A}_r$, $r \ge 2$, in variables $(({}^j w_s : s \in \mathbf{b}) : j = 1, ..., n)$, $\mathbf{b} := \mathbf{b}_r$, as $F(({}^j w_s : s \in \mathbf{b}) : j = 1, ..., n) = f \circ \sigma(({}^j w_s : s \in \mathbf{b}) : j = 1, ..., n)$, where $\sigma(({}^j w_s : s \in \mathbf{b}) : j = 1, ..., n) = ({}^j z : j = 1, ..., n)$ is a bijective mapping. For U open in $\mathcal{A}_r^n$ and $F : U \to \mathcal{A}_r$ we can write F in the form $F = \sum_{s \in \mathbf{b}} F_s s$, where $F_s \in \mathbf{R}$ for each $s \in \mathbf{b}$, $F_{vs} := vF_s$ for each $v \in \mathbf{R}$.

2.2.1. Proposition. *Let $g : U \to \mathcal{A}_r^m$, $r \ge 3$, and $f : W \to \mathcal{A}_r^n$ be two superdifferentiable functions on U and W respectively such that $g(U) \supset W$, U is open in $\mathcal{A}_r^k$, W is open in $\mathcal{A}_r^m$, $k, n, m \in \mathbf{N}$. Then the composite function $f \circ g(z) := f(g(z))$ is superdifferentiable on $V := g^{-1}(W)$ and*
$(Df \circ g(z)).h = (Df(g)).((Dg(z)).h)$
for each $z \in V$ and each $h \in \mathcal{A}_r^k$, where f and g are simultaneously $(z,\tilde{z})$, or z, or $\tilde{z}$-superdifferentiable and hence $f \circ g$ is of the same type of superdifferentiability.

Proof. Since g is superdifferentiable, then g is continuous and $g^{-1}(W)$ is open in $\mathcal{A}_r^k$. Then $f \circ g(z+h) - f \circ g(z) = (Df(g))|_{g=g(z)}.(g(z+h) - g(z)) + \epsilon_f(\eta)|\eta|$, where $\eta = g(z+h) - g(z)$, $g(z+h) - g(z) = (Dg(z)).h + \epsilon_g(h)|h|$ (see §2.2). Since Df is $\mathcal{A}_r^m$-additive and $\mathbf{R}$-homogeneous (and continuous) operator on $\mathcal{A}_r^m$, then
$f \circ g(z+h) - f \circ g(z) = (Df(g))|_{g=g(z)}.((Dg(z)).h) + \epsilon_{f\circ g}(h)|h|$, where
$\epsilon_{f\circ g}(h)|h| := \epsilon_f((Dg(z)).h + \epsilon_g(h)|h|)|(Dg(z)).h + \epsilon_g(h)|h||$
$+[(Df(g))|_{g=g(z)}.(\epsilon_g(h))]|h|)$,
$|(Dg(z)).h + \epsilon_g(h)|h|| \le [\|Dg(z)\| + |\epsilon_g(h)|]|h|$, hence
$|\epsilon_{f\circ g}(h)| \le |\epsilon_f((Dg(z)).h + \epsilon_g(h)|h|)|[\|Dg(z)\| + |\epsilon_g(h)|] + \|(Df(g))|_{g=g(z)}\||\epsilon_g(h)|$
and inevitably $\lim_{h\to 0} \epsilon_{f\circ g}(h) = 0$. Moreover, $\epsilon_{f\circ g}(h)$ is continuous in h, since ϵ_g and ϵ_f are continuous functions, Df and Dg are continuous operators. Evidently, if $\partial_{\tilde{z}} f = 0$ and $\partial_{\tilde{z}} g = 0$ on domains of f and g respectively, then $\partial_{\tilde{z}} f \circ g = 0$ on V, since $D = \partial_z + \partial_{\tilde{z}}$.

2.3. Proposition. *A function $f : U \to \mathcal{A}_r$ is z-superdifferentiable at a*

point $a \in U$ if and only if F is Frechét differentiable at a and $\partial_{\tilde z} f(z)|_{z=a} = 0$. If f is z-superdifferentiable on U, then f is z-represented on U. If $f'(a)$ is right superlinear on the superalgebra $\mathcal{A}_r^n$, then f is z-superdifferentiable at $a \in U$ if and only if F is Frechét differentiable at $a \in U$ and satisfies the following equations:

$$(1) \quad (\partial F_{ps}/\partial\ {}^j w_p) = ((ps)p^*)^*((qs)q^*)(\partial F_{qs}/\partial\ {}^j w_q), \quad \textit{for each } p, q, s \in \mathbf{b}$$

or shortly:

$$(2) \quad \partial F/\partial\ {}^j w_1 = (\partial F/\partial\ {}^j w_q)q^*$$

for each $q \in \hat{b}_r$ and each $j = 1, ..., n$. A $(z, \tilde z)$-superdifferentiable function f at $a \in U$ is z-superdifferentiable at $a \in U$ if and only if $D_{\tilde z} f(z, \tilde z)|_{z=a} = 0$.

Proof. For each canonical closed compact set U in $\mathcal{A}_r$ the set of all polynomial by z functions is dense in the space of all continuous on U Frechét differentiable functions on $Int(U)$.

As usually a set A having structure of an additive group and having distributive multiplications of its elements on Cayley-Dickson numbers $z \in \mathcal{A}_v$ from the left and from the right is called a vector space over $\mathcal{A}_v$. In such sence it is the $\mathbf{R}$-linear space and also the left and right module over $\mathcal{A}_v$. For two vector spaces A and B over $\mathcal{A}_v$ consider their ordered tensor product $A \otimes B$ over $\mathcal{A}_v$ consisting of elements $a \otimes b := (a, b)$ such that $a \in A$ and $b \in B$, $\alpha(a, b) = (\alpha a, b)$ and $(a, b)\beta = (a, b\beta)$ for each $\alpha, \beta \in \mathcal{A}_v$, $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ for each $a_1, a_2 \in A$ and $b_1, b_2 \in B$. In the aforementioned respect $A \otimes B$ is the $\mathbf{R}$-linear space and at the same time left and right module over $\mathcal{A}_v$. Then $A \otimes B$ has the structure of the vector space over $\mathcal{A}_v$. By induction consider tensor products $\{C_1 \otimes C_2 \otimes ... \otimes C_n\}_{q(n)}$, where $C_1, ..., C_n \in \{A, B\}$, $q(n)$ indicates on the order of tensor multiplications in $\{*\}$. For two $\mathcal{A}_v$-vector spaces V and W their direct sum $V \oplus W$ is the $\mathcal{A}_v$-vector space consisting of all elements (a, b) with $a \in V$ and $b \in W$ such that $\alpha(a, b) = (\alpha a, \alpha b)$ and $(a, b)\beta = (a\beta, b\beta)$ for each α and $\beta \in \mathcal{A}_v$. Therefore, the direct sum of all different tensor products $\{C_1 \otimes C_2 \otimes ... \otimes C_n\}_{q(n)}$, which are $\mathbf{R}$-linear spaces and left and right modules over $\mathcal{A}_v$, provides the minimal tensor space $T(A, B)$ generated by A and B.

Operators ∂_z and $\partial_{\tilde z}$ are uniquely defined on $C^\omega_z(U, \mathcal{A}_r)$ and $C^\omega_{\tilde z}(U, \mathcal{A}_r)$, hence they are unique on the tensor space $T(C^\omega_z(U, \mathcal{A}_r), C^\omega_{\tilde z}(U, \mathcal{A}_r))$, which is dense in $C^\omega_{z,\tilde z}(U, \mathcal{A}_r)$, since $C^\omega_{z,\tilde z}(U, \mathcal{A}_r) := C^\omega_{{}_1z,\ {}_2z}(U^2, \mathcal{A}_r)|_{\ {}_1z=z,\ {}_2z=\tilde z}$. Therefore, operators ∂_z and $\partial_{\tilde z}$ are uniquely defined on $C^\omega_{z,\tilde z}(U, \mathcal{A}_r)$.

If there is a product fg of two phrases f and g from $C^{\omega}_{z,\tilde{z}}(U,\mathcal{A}_r)$, then if it is reduced to a minimal phrase ξ, then it is made with the help of $z^nz^m=z^{n+m}$ and $\tilde{z}^n\tilde{z}^m=\tilde{z}^{n+m}$ and identities for constants in $\mathcal{A}_r$, since no any shortening related with their permutation $z\tilde{z}=\tilde{z}z$ or substitution of z on $\tilde{z}$ or $\tilde{z}$ on z, for example, using identity $\tilde{z}=l(zl^*)$ is not allowed in $C^{\omega}_{z,\tilde{z}}(U,\mathcal{A}_r)$ in accordance with our convention in §2.1, since $C^{\omega}_{z,\tilde{z}}(U,\mathcal{A}_r):=C^{\omega}_{{}_1z,\ {}_2z}(U^2,\mathcal{A}_r)|_{{}_1z=z,\ {}_2z=\tilde{z}}$ and in $C^{\omega}_{{}_1z,\ {}_2z}(U^2,\mathcal{A}_r)$ variables ${}_1z$ and ${}_2z$ do not commute, ${}_1z$ and ${}_2z$ are different variables which are not related. Therefeore, $\partial_z\xi.h=(\partial_zf.h)g+f(\partial_zg.h)$ and $\partial_{\tilde{z}}\xi.h=(\partial_{\tilde{z}}f.h)g+f(\partial_{\tilde{z}}g.h)$, hence ∂_z and $\partial_{\tilde{z}}$ are correctly defined. In particular functions of the form of series $f=\sum\{{}_{l_1}f...{}_{l_t}f\}_{q(t)}$ converging on U together with its superdifferential on $Int(U)$ such that each ${}_lf$ is superlinearly z-superdifferentiable on $Int(U)$ relative to the superalgebra $\mathcal{A}_r$ is dense in the $\mathbf{R}$-linear space of z-superdifferentiable functions g on U, since $(Dg(z)).h$ is continuous by (z,h). We can use δ-approximation for each $\delta>0$ of $Dg(z).h$ on a sufficiently small open subset V in U such that $z\in V$ by functions ζ_n polynomial in z and $\mathbf{R}$-homogeneous $\mathcal{A}_r^n$-additive in h and partition of unity in U by C^{ω}_z-functions and then consider functions ξ_n with ${\xi_n}'$ corresponding to ζ_n, since $\epsilon(h)$ is continuous at 0 for each $z\in U$ and for each canonical closed compact subset W in U from each open covering we can choose a finite subcovering of W.

From Conditions 2.2.$(2-7)$ it follows, that the z-superdifferentiability conditions are defined uniquely on space of polynomials. In view of Conditions 2.2.$(1-7)$ the z-superdifferentiability of a polynomial or a converging series P on U means that it is expressible through a sum or a converging series of products of ${}^jz$ and constants from $\mathcal{A}_r$. Therefore, each z-superdifferentiable function f on U is the equivalence class of all Cauchy sequences from $C^{\omega}_z(U,\mathcal{A}_r)$ converging to f relative to the C^1-uniformity, since $(D*).h:C^1\to C^0$ is continuous for each $h\in\mathcal{A}_r^n$.

Suppose that f is z-superdifferentiable at a point a. To each $f'(z)$ there corresponds a $\mathbf{R}$-linear operator on the Euclidean space $\mathbf{R}^{\mathbf{2^rn}}$. Moreover, we have the distributivity and associativity laws for $(Df)(z;h)$ relative to the right multiplication on elements $\lambda\in\mathcal{A}_r$ (see §2.1, 2.2). Then $f(a+h)-f(a)=\partial_af(a).h+\epsilon(h)|h|$ and $\partial_{\tilde{z}}f(z)|_{z=a}=0$, since generally $f(a+h)-f(a)=(\partial_af(a)).h+(\partial_{\tilde{a}}f(a)).h+\epsilon(h)|h|$, where $\epsilon(h)$ is continuous by h and $\epsilon(0)=0$. Vice versa, if F is Frechét differentiable and $\partial_{\tilde{z}}f(z)|_{z=a}=0$, then expressing w_ss for each $s\in\mathbf{b}_r$ through linear combinations of z (with multiplication on constants from $\mathcal{A}_r$ on the left and on the right) with constant coefficients we

get the increment of f as above.

Consider now the particular case, when f' is right superlinear on the superalgebra $\mathcal{A}_r^n$ and $\partial_{\tilde{z}} f(z)|_{z=a} = 0$. In this case $f'(a)$ is right $\mathcal{A}_r$-linear. Using the definition of the z-superderivative and that there is a bijective correspondence between z and $(({}^jw_s : s \in \mathbf{b}_r) : j = 1, ..., n)$, ${}^jw_s \in \mathbf{R}$ for each $s \in \mathbf{b}_r$, $j = 1, ..., n$, we consider a function $f = f(z)$ and $F(({}^jw_s : s \in \mathbf{b}_r) : j = 1, ..., n) = f \circ \sigma$, where f is z-superdifferentiable by z, hence F is Frechét differentiable by $(({}^jw_s : s \in \mathbf{b}_r) : j = 1, ..., n)$ and we obtain the expressions:

$$\partial F/\partial {}^jw_s = (\partial F/\partial {}^jz).(\partial {}^jz/\partial {}^jw_s),$$

since $\partial {}^jz/\partial {}^kw_s = 0$ and $\partial {}^j\tilde{z}/\partial {}^kw_s = 0$ and $\partial f(z)/\partial {}^j\tilde{z}|_{z=a} = 0$ for each $k \neq j$. From $\partial {}^jz/\partial {}^jw_s = s$ for each $s \in \mathbf{b}_r$ we get Equations (2), since $ps = -sp$, $F_{ps} = -F_{sp}$ and $pp^* = 1$ for each $p \neq s \in \hat{b}$. Using the equality $F = \sum_{s \in \mathbf{b}} F_s s$ we get Equations (1) from the latter equations, since qf is right superlinearly superdifferentiable together with f for each $q \in \mathbf{b}$, $((ps)p^*)^*((qs)q^*) \in \{-1, 1\} \subset \mathbf{R}$ for each $p, q, s \in \mathbf{b}$.

Let now F be Frechét differentiable at a and let F be satisfying Conditions (1). Then $f(z) - f(a) = \sum_{j=1}^n \sum_{s \in \mathbf{b}} (\partial F/\partial {}^jw_s)\Delta {}^jw_s + \epsilon(z-a)|z-a|$, where $\Delta({}^jw_s : s \in \mathbf{b}) = \sigma^{-1}({}^jz) - \sigma({}^ja)$ for each $j = 1, ..., n$. From Conditions (2) equivalent to (1) we get
$f(z) - f(a) = \sum_{j=1}^n \sum_{s \in \mathbf{b}} (\partial F/\partial {}^jw_1) s \Delta {}^jw_s + \epsilon(z-a)|z-a| =$
$\sum_{j=1}^n (\partial F/\partial {}^jw_1)\Delta {}^jz + \epsilon(z-a)|z-a|$,
where ϵ is a function continuous at 0 and $\epsilon(0) = 0$. Therefore, f is superdifferentiable by z at a such that $f'(a)$ is right superlinear, since $\partial F/\partial {}^jw_s$ are real matrices and hence $f'(a).({}_1h\lambda_1 + {}_2h\lambda_2) = (f'(a) {}_1h)\lambda_1 + (f'(a) {}_2h)\lambda_2$ for each λ_1 and $\lambda_2 \in \mathcal{A}_r$ and each ${}_1h$ and ${}_2h \in \mathcal{A}_r^n$.

The last statement of this proposition follows from Definition 2.2.

2.3.1. Notation. If $f : U \to \mathcal{A}_r$ is either z-superdifferentiable or $\tilde{z}$-superdifferentiable at $a \in U$ or on U, then we can write also $D_{\tilde{z}}$ instead of $\partial_{\tilde{z}}$ and D_z instead of ∂_z at $a \in U$ or on U respectively in situations, when it can not cause a confusion, where U is open in $\mathcal{A}_r^n$.

2.4. Corollary. *Let f be a continuously superdifferentiable function by z with a right superlinear superdifferential on the superalgebra $\mathcal{A}_r^n$, $r \geq 3$, in an open subset U in $\mathcal{A}_r^n$ and let F be twice continuously differentiable by $(({}^jw_s : s \in \mathbf{b}_r) : j = 1, ..., n)$ in U, then each component F_s of F is the harmonic functions by pairs of variables $({}^jw_p, {}^jw_q)$ for each $p \neq q \in \mathbf{b}_r$*

namely:

$$(1)\quad \triangle_{\ ^jw_p,\ ^jw_q} F_s = 0,$$

for each $j = 1, .., n$, *where* $\triangle_{\ ^jw_p,\ ^jw_q} F_s := \partial^2 F_s/\partial\ ^jw_p^2 + \partial^2 F_s/\partial\ ^jw_q^2$.

Proof. From Equations 2.3.(1) and in view of the twice continuous differentiability of F it follows, that $(\partial^2 F_s/\partial\ ^jw_p^2) = (\partial^2 F_{(sp^*)q}/\partial\ ^jw_p \partial\ ^jw_q) = \partial^2 F_{(((sp^*)q)p^*)q}/\partial\ ^jw_q^2) = -(\partial^2 F_s/\partial\ ^jw_q^2)$, since $F_{vs} = vF_s$ for each $v \in \mathbf{R}$ and each $s \in \mathbf{b}_r$, $p \ne q \in \mathbf{b}_r$ and hence $p^*q \in \hat{b}_r$, $t^2 = -1$ for each $t \in \hat{b}_r$, $p^* = -p$ for each $p \in \hat{b}_r$, $pq = -qp$ for each $p \ne q \in \hat{b}_r$.

2.5. Note and Definition. Let U be an open subset in $\mathcal{A}_r$ and let $f : U \to \mathcal{A}_r$, $r \ge 3$, be a function defined on U such that

$$(i)\quad f(z, \tilde{z}) = \{f^1(z, \tilde{z})...f^j(z, \tilde{z})\}_{q(j)},$$

where each function $f^s(z, \tilde{z})$ is presented by a Laurent series

$$(ii)\quad f^s(z, \tilde{z}) = \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} (f^s_{n,m}(z - \zeta)^n)(\tilde{z} - \tilde{\zeta})^m$$

converging on U, where $f^s_{n,m} \in \mathcal{A}_r$, $z \in U$, $\zeta \in \mathcal{A}_r$ is a marked point, n and $m \in \mathbf{Z}$, if $n_0 < 0$ or $m_0 < 0$, then $\zeta \notin U$. Consider the case $f^s_{-1,m} = 0$ for each s and m. The case with terms $f^s_{-1,m} \ne 0$ will be considered later.

Let $[a, b]$ be a segment in $\mathbf{R}$ and $\gamma : [a, b] \to \mathcal{A}_r$ be a continuous function. Consider a partitioning P of $[a, b]$, that is, P is a finite subset of $[a, b]$ consisting of an increasing sequence of points $a = c_0 < ... < c_k < c_{k+1} < ... < c_t = b$, then the norm of P is defined as $|P| := \max_k (x_{k+1} - x_k)$ and the P-variation of γ as $v(\gamma; P) := \sum_{k=0}^{t-1} |\gamma(c_{k+1}) - \gamma(c_k)|$, where $t = t(P) \in \mathbf{N}$. The total variation (or the length) of γ is defined as $V(\gamma) = \sup_P v(\gamma; P)$. Suppose that γ is rectifiable, that is, $V(\gamma) < \infty$. For f having decomposition $(2.5.i, ii)$ with $f^s_{-1,m} = 0$ for each s and m and a rectifiable path $\gamma : [a, b] \to U$ we define a (noncommutative) Cayley-Dickson algebra line integral. Consider more general situation.

Let $f : U \to \mathcal{A}_r$ be a continuous function, where U is open in $\mathcal{A}_r$, f is defined by a continuous function $\xi : U^2 \to \mathcal{A}_r$ such that

(1) $\xi(\ _1z,\ _2z)|_{\ _1z=z,\ _2z=\tilde{z}} = f(z, \tilde{z})$

or shortly $f(z)$ instead of $f(z, \tilde{z})$, where $\ _1z$ and $\ _2z \in U$. Let also $g : U^2 \to \mathcal{A}_r$ be a continuous function, which is $\ _1z$-superdifferentiable such that

(2) $(\partial g(\ _1z,\ _2z)/\partial\ _1z).1 = \xi(\ _1z,\ _2z)$ on U^2. Then put

(3) $\hat{f}(z,\tilde{z}).h := \hat{f}_z(z,\tilde{z}).h := [(\partial g(\ _1z,\ _2z)/\partial\ _1z).h]|_{\ _1z=z,\ _2z=\tilde{z}}$ for each $h \in \mathcal{A}_r$. Shortly we can write these as $(\partial g(z,\tilde{z})/\partial z).1 = f(z,\tilde{z})$ and $\hat{f}_z(z,\tilde{z}).h := \hat{f}(z).h := (\partial g(z,\tilde{z})/\partial z).h$. If the following limit exists

$$(4) \quad \int_\gamma f(z,\tilde{z})dz := \lim_P I(f,\gamma;P), \text{ where}$$

$$(5) \quad I(f,\gamma;P) := \sum_{k=0}^{t-1} \hat{f}(z_{k+1},\tilde{z}_{k+1}).(\Delta z_k),$$

where $\Delta z_k := z_{k+1} - z_k$, $z_k := \gamma(c_k)$ for each $k = 0,...,t$, then we say that f is line integrable along γ by the variable z. Analogously we define $\int_\gamma f(z,\tilde{z})d\tilde{z}$ with $(\partial g(z,\tilde{z})/\partial\tilde{z}).1 = f(z,\tilde{z})$, $\hat{f}_{\tilde{z}}(z,\tilde{z}) := (\partial g(z,\tilde{z})/\partial\tilde{z}).h$, where $g(\ _1z,\ _2z)$ is $\ _2z$-superdifferentiable.

Remark. In view of Definitions 2.1, 2.2 and Proposition 2.3 Conditions 2.5.$(1-3)$ are correct, for example, the taking of functions ξ and g in the $(\ _1z,\ _2z)$-representation is sufficient.

This definition is justified by the following lemma and proposition.

2.5.1. Lemma. *Let $f : U \to \mathcal{A}_r$ satisfy Conditions* 2.2.(1, 2, 5)*, where U is an open subset in $\mathcal{A}_r$. Then conditions*

(1) $\partial f(z)/\partial\bar{z}_{2j,2j+1} = 0$ *for each* $j = 0,1,...,2^{r-1}-1$ *and* $z \in U$,
where $z_{2j,2j+1} := w_s + w_p s^* p$ with $s = i_{2j}$, $p = i_{2j+1}$, are equivalent with $\partial_{\tilde{z}} f(z) = 0$ for each $z \in U$.

Proof. Since $(z+h)^* - z^* = h^*$, $((\lambda h)^*)^* = \lambda h$ for each z and $h \in \mathcal{A}_r^n$ and each $\lambda \in \mathcal{A}_r$, $(h_1 + h_p p)^* = h_1 - h_p p$ for each $p \in \hat{b}$, $(s(h_s + h_p s^* p))^* = (h_s - h_p s^* p)s^*$ for each $s \ne p \in \hat{b}$, $(\partial z/\partial z).h = h$, $(\partial\tilde{z}/\partial\tilde{z}).h = \tilde{h}$, $\partial z/\partial\tilde{z} = 0$ and $\partial\tilde{z}/\partial z = 0$, where $h_p \in \mathbf{R}$ for each $p \in \mathbf{b}$, hence
$(\partial f(z)/\partial z_{2j,2j+1}).h = (\partial f(z)/\partial z).(sh)|_{h\in\mathbf{R}\oplus s^*p\mathbf{R}}$ and
$(\partial f(z)/\partial\bar{z}_{2j,2j+1}).h = (\partial f(z)/\partial\tilde{z}).(sh)|_{h\in\mathbf{R}\oplus s^*p\mathbf{R}}$, where $s = i_{2j}$ and $p = i_{2j+1}$, then
$\partial z/\partial\bar{z}_{2j,2j+1} = 0$, $\partial\tilde{z}/\partial z_{2j,2j+1} = 0$ and
$(\partial\bar{z}_{2j,2j+1}/\partial\bar{z}_{2j,2j+1}).h = h$, $(\partial z_{2j,2j+1}/\partial z_{2j,2j+1}).h = h$ for each $h \in \mathbf{R}\oplus s^*p\mathbf{R}$.
In view of Proposition 2.2.1 from $\partial_{\tilde{z}} f(z) = 0$ on U and Conditions 2.2.(1, 2, 5) it follows, that for each j:
$\partial f(z)/\partial\bar{z}_{2j,2j+1} = (\partial f(z)/\partial\tilde{z}).(\partial\tilde{z}/\partial\bar{z}_{2j,2j+1}) + (\partial f(z)/\partial z).(\partial z/\partial\bar{z}_{2j,2j+1}) = 0$, since $\partial_{\tilde{z}} f(z) = 0$ and $\partial z/\partial\bar{z}_{2j,2j+1} = 0$. Generally for f its derivative $f'(z)$ need not be right (or left) superlinear on $\mathcal{A}_r$. Then $\partial f(z)/\partial\bar{z}_{2j,2j+1}$ are **R**-

homogeneous additive operators on the $\mathbf{R}$-linear subspace $(\mathbf{R} \oplus s^*p\mathbf{R})$ of $\mathcal{A}_r$.

Let f satisfy 2.2.$(1, 2, 5)$ and (1), then

$(Df(z)).h = \sum_{s\in\mathbf{b}}(Df(z)).h_s s = \sum_{s\in\mathbf{b}}(\partial f(z)/\partial w_s)h_s$,
since $\partial z/\partial w_s = s$ for each $s \in \mathbf{b}$, where $h = \sum_{s\in\mathbf{b}} h_s s \in \mathcal{A}_r$, $h_s \in \mathbf{R}\ \forall\ s \in \mathbf{b}$. From
$(\partial f(z)/\partial w_s)h_s + (\partial f(z)/\partial w_p)h_p = (\partial f(z)/\partial z_{2j,2j+1}).(h_s + s^*ph_p)$
for each $s = i_{2p}$ and $p = i_{2j+1}$, since
$(\partial f(z)/\partial w_s) = (\partial f(z)/\partial z_{2j,2j+1}).1 + (\partial f(z)/\partial \bar{z}_{2j,2j+1}).1$,
$(\partial f(z)/\partial w_p) = (\partial f(z)/\partial z_{2j,2j+1} - \partial f(z)/\partial \bar{z}_{2j,2j+1}).(s^*p)$
and $\partial f(z)/\partial \bar{z}_{2j,2j+1} = 0$, it follows that $(Df(z)).h = (\partial f(z)/\partial z).h$, since by Proposition 2.2.1 and Conditions (1)
$(\partial f(z)/\partial \tilde{z}).(h_s s + h_p p) = (\partial f(z)/\partial \bar{z}_{2j,2j+1}).(h_s + h_p s^*p) = 0$ for each j
and $(\partial f/\partial \tilde{z}).h = \sum_{j=0}^{2^{r-1}-1}(\partial f/\partial \tilde{z}).(i_{2j}(h_{2j} + h_{2j+1}i_{2j}^* i_{2j+1}))$, since $h_s \in \mathbf{R}$ for each $s \in \mathbf{b}$.

2.6. Proposition. *Let f be a function as in §2.5 and suppose that there are two constants r and R such that the Laurent series $(2.5.i, ii)$ converges in the set $B(a, r, R, \mathcal{A}_m) := \{z \in \mathcal{A}_m : r \le |z - a| \le R\}$ for each $s = 1, ..., j$, let also γ be a rectifiable path contained in $U \cap B(a, r', R', \mathbf{H})$, where $r < r' < R' < R$, $m \ge 3$. Then the Cayley-Dickson algebra line integral exists.*

Proof. At first mention that $\mathcal{A}_m$ is the normed algebra such that $|\xi\eta| \le |\xi||\eta|$ for each ξ and $\eta \in \mathcal{A}_m$. This can be proved by induction starting from $\mathbf{C}$ and using the doubling procedure. Suppose $m \ge 2$ and $\mathcal{A}_{m-1}$ is the normed algebra, then $|(a, b)(c, d)| = (|ac - d^*b|^2 + |da + bc^*|^2)^{1/2} \le (|ac|^2+|d^*b|^2+|da|^2+|bc^*|^2)^{1/2} \le (|a|^2+|b|^2)^{1/2}(|c|^2+|d|^2)^{1/2} = |(a, b)||(c, d)|$, where $a, b, c, d \in \mathcal{A}_{m-1}$, (a, b) and $(c, d) \in \mathcal{A}_m$. Since each f^s converges in $B(a, r, R, \mathcal{A}_m)$, then

$$\overline{\lim}_{n+m>0}|f^s_{n,m}|^{1/(n+m)}R \le 1, \text{ hence}$$

$$\|f\|_\omega := \prod_{s=1}^{j}(\sup_{n+m<0}|f^s_{n,m}|r^{n+m}, \sup_{n+m\ge0}|f^s_{n,m}|R^{n+m}) < \infty$$

and inevitably

$$\|f\|_{1,\omega,B(a,r',R',\mathcal{A}_m)} := \prod_{s=1}^{j}[(\sum_{n+m<0}|f^s_{n,m}|r'^{n+m}) + (\sum_{n+m>0}|f^s_{n,m}|R'^{n+m})] < \infty.$$

For each locally z-analytic function f in U and each z_0 in U there exists a ball of radius $r > 0$ with center z_0 such that f has a decomposition analogous to $(2.5.i, ii)$ in this ball with all n nonnegative. Consider two z-locally analytic functions f and q on U such that f and q noncommute. Let $f^0 := f$, $q^0 := q$, $q^{-n} := q^{(n)}$, $(\partial (q^n)/\partial z).1 =: q^{n-1}$ and $q^{-k-1} = 0$ for some $k \in \mathbf{N}$, then

(i) $(fq)^1 = f^1 q - f^2 q^{-1} + f^3 q^{-2} + ... + (-1)^k f^{k+1} q^{-k}$. In particular, if $f = az^n$, $q = bz^k$, with $n > 0$, $k > 0$, $b \in \mathcal{A}_m \setminus \mathbf{R}I$, then $f^p = [(n+1)...(n+p)]^{-1} a z^{n+p}$ for each $p \in \mathbf{N}$, $q^{-s} = k(k-1)...(k-s+1) b z^{k-s}$ for each $s \in \mathbf{N}$. Also

(ii) $(fq)^1 = f q^1 - f^{-1} q^2 + f^{-2} q^3 + ... + (-1)^n f^{-n} q^{n+1}$, when $f^{-n-1} = 0$ for some $n \in \mathbf{N}$. Apply (i) for $n \ge m$ and (ii) for $n < k$ to solve the equation $(\partial g(z, \tilde{z})/\partial z).1 = f(z, \tilde{z})$ for each $z \in U$. If f and q have series converging in $Int(B(z_0, r, \mathcal{A}_m))$, then these formulas show that there exists a z-analytic function $(fq)^1$ with series converging in $Int(B(0, r, \mathcal{A}_m))$, since $\lim_{n\to\infty} (nr^n)^{1/n} = r$, where $0 < r < \infty$. Applying this formula by induction to products of polynomials $\{P_1 ... p_n\}_{q(n)}$ and converging series, we get g. Since f is locally analytic, then g is also locally analytic. Therefore, for each locally z-analytic function f there exists the operator $\hat{f}$. Considering a function G of real variables corresponding to g we get that due to Lemma 2.5.1 all solutions g differ on constants in $\mathcal{A}_m$, since $\partial g/\partial w_s + (\partial g/\partial w_p).(s^* p) = 0$ for each $s = i_{2j}$, $p = i_{2j+1}$, $j = 0, 1, ..., 2^{r-1} - 1$ and $\partial g/\partial w_1$ is unique, hence $\hat{f}$ is unique for f. Therefore,

$$(1) \quad |\{f^s_{n,m} (z_{j+1} - a)^k (\Delta z_j)(z_{j+1} - a)^{n-k} (\tilde{z}_{j+1} - \tilde{a})^m\}_{q(n+m+1)}| \le$$
$$|f^s_{n,m}||(z_{j+1} - a)^n (\tilde{z}_{j+1} - \tilde{a})^m||\Delta z_j|.$$

From Equation (1) it follows, that $|I(f, \gamma; P)| \le \|f\|_{1,\omega,B(a,r',R',\mathbf{H})} \upsilon(\gamma; P)$, for each P, and inevitably

$$(2) \quad |I(f, \gamma; P) - I(f, \gamma; Q)| \le w(\hat{f}; P) V(\gamma)$$

for each $Q \supset P$, where

$$(3) \quad w(\hat{f}; P) := \max_{(z,\zeta \in \gamma([c_j, c_{j+1}]))} \{\|\hat{f}(z) - \hat{f}(\zeta)\| : \quad z_j = \gamma(c_j), c_j \in P\},$$

$\|\hat{f}(z) - \hat{f}(\zeta)\| := \sup_{h \ne 0} |\hat{f}(z).h - \hat{f}(\zeta).h|/|h|$. Since $\lim_{n\to\infty} (n)^{1/n} = 1$, then $\lim_P \omega(\hat{f}, P) = 0$. From $\lim_P w(\hat{f}; P) = 0$ the existence of $\lim_P I(f, \gamma; P)$ now follows.

2.7. Theorem. *Let γ be a rectifiable path in U, then the Cayley-Dickson algebra $\mathcal{A}_r$, $r \geq 3$, line integral has a continuous extension on the space $C^0_b(U, \mathcal{A}_r)$ of bounded continuous functions $f : U \to \mathcal{A}_r$. This integral is an $\mathbf{R}$-linear and left-$\mathcal{A}_r$-linear and right-$\mathcal{A}_r$-linear functional on $C^0_b(U, \mathcal{A}_r)$.*

Proof. Since γ is continuous and $[a, b]$ is compact, then there exists a compact canonical closed subset V in $\mathcal{A}_r$, that is, $cl(Int(V)) = V$, such that $\gamma([a, b]) \subset V \subset U$. Let $f \in C^0_b(U, \mathcal{A}_r)$, then in view of the Stone-Weierstrass theorem for a function $F(w_s : s \in \mathbf{b}) = f \circ \sigma(w_s : s \in \mathbf{b})$ and each $\delta > 0$ there exists a polynomial T such that $\|F - T\|_0 < \delta$, where $\|f\|_0 := \sup_{z\in U} |f(z)|$. This polynomial takes values in $\mathcal{A}_r$, hence it has the form: $T = \sum_{s\in\mathbf{b}} T_s s$, where $T_s : U \to \mathbf{R}$. There are relations $zp = \sum_{s\in\mathbf{b}} w_s sp$, $sp = -ps$ for each $s \neq p \in \hat{b}$, consequently, $zp = -w_p + \sum_{s\in\mathbf{b};s\neq p} w_s sp$, $(zp)^* = p^*z^* = -w_p + \sum_{s\in\mathbf{b};s\neq p} w_s(sp)^* = -w_p - \sum_{s\in\mathbf{b},s\neq p} w_s sp$, since $p^* = -p$ for each $p \in \hat{b}$. Then $w_1 = (z + \tilde{z})/2$ and $w_p = (p\tilde{z} - zp)/2$ for each $p \in \hat{b}$, where we use the identity $\tilde{z} = (2^r - 2)^{-1}\{-z + \sum_{s\in\hat{b}} s(zs^*)\}$ (see §2.1). Write $F = \sum_{s\in\mathbf{b}} F_s s$, where $F_s \in \mathbf{R}$ for each $s \in \mathbf{b}$, then the application of the Stone-Weierstrass theorem by real variables $(w_s : s \in \mathbf{b})$ expressed through z and $s \in \mathbf{b}$ with real constant multipliers gives that the $\mathbf{R}$-linear space of functions given by Equations $(2.5.i, ii)$ is dense in $C^0_b(U, \mathcal{A}_r)$.

Consider a function $g(z)$ on U. Let $g(z)$ be superdifferentiable by z. Consider a space of all such that g on U for which $(Dg(z)).s$ is a bounded continuous function on U for each $s \in \mathbf{b}$, it is denoted by $C^1_b(U, \mathcal{A}_r)$ and it is supplied with the norm $\|g\|_{C^1_b} := \|g\|_{C^0_b} + \sum_{s\in\mathbf{b}} \|(Dg(z).s\|_{C^0_b}$, where $\|g\|_{C^0_b} := \sup_{z\in U} |g(z)|$, hence $(Dg(z)).h \in C^0_b(U \times B(0, 0, 1, \mathcal{A}_r), \mathcal{A}_r)$, where $h \in B(0, 0, 1, \mathcal{A}_r)$. Therefore, there exists a positive constant C such that

$$(1) \quad \sup_{h\neq 0} |(Dg(z)).h|/|h| \leq C \sum_{s\in\mathbf{b}} \|(Dg(z)).s\|_{C^0_b},$$

since $h = \sum_{s\in\mathbf{b}} h_s s$ for each $h \in \mathcal{A}_r$ and $Dg(z)$ is $\mathbf{R}$-linear and $(Dg(z)).({}^1h + {}^2h) = (Dg(z)).{}^1h + (Dg(z)).{}^2h$ for each ${}^1h$ and ${}^2h \in \mathcal{A}_r$, where h_s is a real number for each $s \in \mathbf{b}$, $G(w_s : s \in \mathbf{b}) := g \circ \sigma(w_s : s \in \mathbf{b})$ is Frechét differentiable on an open subset $U_\sigma \subset \mathbf{R}^{\mathbf{2^r}}$ such that $\sigma(U_\sigma) = U$.

In §2.6 it was shown that the equation $(\partial g(z, \tilde{z})/\partial z).1 = f(z, \tilde{z})$ has a solution in a class of locally z-analytic functions on U. The subset $C^\omega(U, \mathcal{A}_r)$ is dense in the uniform space $C^0_b(U, \mathcal{A}_r)$.

If $g = \{g^1...g^j\}_{q(j)}$ is a product of functions $g^s \in C^1_b(U, \mathcal{A}_r)$, then $(Dg(z)).h = \sum_{v=1}^j \{g^1(z)...g^{v-1}(z)[(Dg^v(z)).h]g^{v+1}(z)...g^j(z)\}_{q(j)}$ for each $h \in \mathcal{A}_r$. Con-

sider the space $\hat{C}^0_b(U,\mathcal{A}_r) := \{(Dg(z)).s : s \in \mathbf{b}\}$. It has an embedding ξ into $C^0_b(U,\mathcal{A}_r)$ and $\|g\|_{C^1_b} \ge \sum_{s\in\mathbf{b}} \|(Dg(z)).s\|_{C^0_b}$. In view of Inequality (1) the completion of $\hat{C}^0_b(U,\mathcal{A}_r)$ relative to $\|*\|_{C^0_b(U,\mathcal{A}_r)}$ coincides with $C^0_b(U,\mathcal{A}_r)$.

Let $\{f^v : \quad v \in \mathbf{N}\}$ be a sequence of functions having decomposition $(2.5.i, ii)$ and converging to f in $C^0_b(U,\mathcal{A}_r)$ relative to the metric $\rho(f,q) := \sup_{z\in U} |f(z) - q(z)|$ such that $f^v = \xi((Dg^v(z)).s : s \in \mathbf{b})$ for some $g^v \in C^1_b(U,\mathcal{A}_r)$. Relative to this metric $C^0_b(U,\mathcal{A}_r)$ is complete. We have the equality

$$\partial(\int_0^q F(\phi h_s : s \in \mathbf{b}))/\partial q = F(w_s : s \in \mathbf{b})$$

for each continuous function F on U_σ, where $w_s = w_{0,s} + qh_s$ for each $s \in \mathbf{b}$, $(w_{0,s} : s \in \mathbf{b}) + \phi(h_s : s \in \mathbf{b}) \in U_\sigma$ for each $\phi \in \mathbf{R}$ with $0 \le \phi < q + \epsilon$, $0 < \epsilon < \infty$, $h_s \in \mathbf{R}$ for each $s \in \mathbf{b}$. Let z_0 be a marked point in V. There exists $R > 0$ such that γ is contained in the interior of the parallelepiped $V := \{z \in \mathcal{A}_r : z = \sum_{s\in\mathbf{b}} w_s s;\ |w_s - w_{0,s}| \le R$ for each $s \in \mathbf{b}\}$.

If V is not contained in U consider a continuous extension of a continuous function F from $V \cap U_0$ on V, where U_0 is a closed subset in U such that $Int(U_0) \supset \gamma$ (about the theorem of a continuous extension see [9]). Therefore, suppose that F is given on V. Then the function $F_1(w_s : s \in \mathbf{b}) := \int_{w_{0,1}}^{w_1} ... \int_{w_{0,t}}^{w_t} F(w_s : s \in \mathbf{b}) dw_1...dw_t$ is in $C^1(V,\mathcal{A}_r)$ (with one sided derivatives on ∂V from inside V), where $t := i_{2^r-1}$. Consider a foliation of V by $(2^r - 1)$-dimensional C^0-manifolds Υ_z such that $\Upsilon_z \cap \Upsilon_{z_1} = \emptyset$ for each $z \ne z_1$, where $z, z_1 \in \gamma$, $\bigcup_{z\in\gamma} \Upsilon_z = V_1$, V_1 is a canonical closed subset in $\mathcal{A}_r$ such that $\gamma \subset V_1 \subset V$. Choose this foliation such that to have decomposition of a Lebesgue measure dV into the product of measures $d\nu(z)$ along γ and $d\Upsilon_z$ for each $z \in \gamma$. In view of the Fubini theorem there exists $\int_V f(w_s : s \in \mathbf{b}) dV = \int_\gamma (\int_{\Upsilon_z} f(z) d\Upsilon_z) d\nu(z)$. If γ is a straight line segment then $\int_\gamma f(z)dz$ is in $L^1(\Upsilon,\mathcal{A}_r)$. Let $U_{\mathbf{R}}$ be a real region in $\mathbf{R^{2^r}}$ corresponding to U in $\mathcal{A}_r$.

Consider the Sobolev space $W^q_2(U_{\mathbf{R}}, \mathbf{R^{2^r}})$ of functions $h : U_{\mathbf{R}} \to \mathbf{R^{2^r}}$ for which $D^\alpha h \in L^2(U_{\mathbf{R}}, \mathbf{R^{2^r}})$ for each $|\alpha| \le q$, where $0 \le q \in \mathbf{Z}$. In view of Theorem 18.1.24 [14] (see also the notation there) if $A \in \Psi^m$ is a properly supported pseudodifferential elliptic operator of order m in the sence that the principal symbol $a \in S^m(T^*(X))/S^{m-1}(T^*(X))$ has an inverse in $S^{-m}(T^*(X))/S^{-m-1}(T^*(X))$, then one can find $B \in \Psi^{-m}$ properly supported such that $BA - I \in \Psi^{-\infty}$, $AB - I \in \Psi^{-\infty}$. One calls B a parametrix for A. In view of Proposition 18.1.21 [14] each $A \in \Psi^m$ can be written as a sum

$A = A_1 + A_0$, where $A_1 \in \Psi^m$ is properly supported and the kernel of A_0 is in C^∞. In particular we can take a pseudodifferential operator with the principal symbol $a(x,\xi) = (b + |\xi|^2)^{s/2}$, where $b > 0$ is a constant and $s \in \mathbf{Z}$, which corresponds to $b+\Delta$ for $s = 1$ up to minor terms, where $\Delta = \nabla^2$ is the Laplacian (see also Theorem 3.2.13 [10] about its parametrix family). For estimates of a solution there may be also applied Theorem 3.3.2 and Corollary 3.3.3 [10] concerning parabolic pseudodifferential equations for our particular case corresponding to $(\partial g(z,\tilde{z})/\partial z).1 = f$ rewritten in real variables.

Due to the Sobolev theorem (see [32, 33]) there exists an embedding of the Sobolev space $W_2^{2^{r-1}+1}(V,\mathcal{A}_r)$ into $C^0(V,\mathcal{A}_r)$ such that

(2) $\|g\|_{C^0} \le C\|g\|_{W_2^{2^{r-1}+1}}$ for each $g \in W_2^{2^{r-1}+1}$, where C is a positive constant independent of g. If $h \in W_2^{k+1}(V,\mathcal{A}_r)$, then $\partial h/\partial w_s \in W_2^k(V,\mathcal{A}_r)$ for each $k \in \mathbf{N}$ and in particular for $k = 2^{r-1}+1$ and each $s \in \mathbf{b}$ (see [32]). On the other hand, $\|h\|_{L^2(V,\mathcal{A}_r)} \le \|h\|_{C^0(V,\mathcal{A}_r)}(2R)^{2^{r-1}}$ for each $h \in L^2(V,\mathcal{A}_r)$. Therefore,

(3) $\|A^{-k}h\|_{W_2^k(V,\mathcal{A}_r)} \le C\|h\|_{C^0(V,\mathcal{A}_r)}(2R)^{k+2^{r-1}}$ for each $k \in \mathbf{N}$, where $C = const > 0$, A is an elliptic pseudodifferential operator such that A^2 corresponds to $(1+\Delta)$. For the estimate below the Gronwall Lemma is used (see, for example, Section 3.3.1 [3]), which reads as follows. Let $\phi(t)$ and $\psi(t)$ be measurable bounded functions, and $\eta(t)$ be a continuous nonnegative function such that

$\phi(t) \le X + \psi(t) + \int_0^t \eta(\tau)\phi(\tau)d\tau$. Then

$\phi(t) \le X\exp[\int_0^t \eta(\tau)d\tau] + \psi(t) + \int_0^t \exp[\int_\tau^t \eta(v)dv]\psi(\tau)\eta(\tau)d\tau$.

Use this lemma for $\phi(t) := |\int_{z\in\{\gamma(v):a\le v\le t\}}(f(z)-q(z))dz|$, $X := C_1\rho(f,q)V(\gamma)$, $\psi(t) = 0$, $\eta(t) = {C'}_2R^{2^r+1}$, where $C_1 > 0$ and ${C'}_2 > 0$ are suitable constants independent of f, q and γ, since $\|(\hat{f}-\hat{q})(z)\| \le \rho(f,q) + \|Im\circ(\hat{f}-\hat{q})(z)\|$ and $\|Im\circ(\hat{f}-\hat{q})(z)\| \le \|\hat{f}-\hat{q}(z)\|$ and $\hat{f}(z).1 = f$ for each $z \in \gamma([a,b])$, $|(\hat{f}-\hat{q})(\gamma(x_{k+1})).(\gamma(x_{k+1})-\gamma(x_k))| \le \|(\hat{f}-\hat{q})(\gamma(x_{k+1}))\||\gamma(x_{k+1})-\gamma(x_k)|$ for each partitioning P: $a = x_0 < ... < x_k < x_{k+1} < ... < x_w = b$, where $Im(z) := (z - l(zl^*))/2$. From Equations 2.5.(1, 2) and Inequalities $(1-3)$ it follows, that there exists $0 < \epsilon < \infty$ such that

$$(4) \quad |I(f-q,\gamma;P)| \le \rho(f,q)V(\gamma)C_1\exp(C_2R^{2^r+2})$$

for each partitioning P of norm $|P|$ less than ϵ, where C_1 and C_2 are positive constants independent of R, f and q. In view of Formulas 2.6.(1, 2)

$\{\int_\gamma f^v(z)dz : v \in \mathbf{N}\}$ is a Cauchy sequence in $\mathcal{A}_r$ and the latter is complete as the metric space. Therefore, there exists $\lim_v \lim_P I(f^v,\gamma;P) = \lim_v \int_\gamma f^v(z)dz$, which we denote by $\int_\gamma f(z)dz$. As in §2.6 we get that all solutions g differ on quaternion constants on each connected component of U, consequently, the functional $\int_\gamma$ is uniquely defined on $C^0_b(U,\mathcal{A}_r)$. The functional $\int_\gamma : C^0_b(U,\mathcal{A}_r) \to \mathcal{A}_r$ is continuous due to Formula (4) and evidently it is $\mathbf{R}$-linear, since $\lambda z = z\lambda$ for each $\lambda \in \mathbf{R}$ and each $z \in \mathcal{A}_r$, that is, $\int_\gamma(\lambda_1 f_1(z) + \lambda_2 f_2(z))dz = \int_\gamma (f_1(z)\lambda_1 + f_2(z)\lambda_2)dz = \lambda_1 \int_\gamma f_1(z)dz + \lambda_2 \int_\gamma f_2(z)dz$ for each λ_1 and $\lambda_2 \in \mathbf{R}$, f_1 and $f_2 \in C^0_b(U,\mathcal{A}_r)$. Moreover, it is left-$\mathcal{A}_r$-linear, that is, $\int_\gamma(\lambda_1 f_1(z) + \lambda_2 f_2(z))dz = \lambda_1 \int_\gamma f_1(z)dz + \lambda_2 \int_\gamma f_2(z)dz$ for each λ_1 and $\lambda_2 \in \mathcal{A}_r$, f_1 and $f_2 \in C^0_b(U,\mathcal{A}_r)$, since $I(f,\gamma;P)$ is left-$\mathcal{A}_r$-linear. If $g_k \in C^1(U,\mathcal{A}_r)$, then $g_k\lambda \in C^1(U,\mathcal{A}_r)$ for each $\lambda \in \mathcal{A}_r$ and $(Dg_k(z)\lambda).h = (Dg_k(z).h)\lambda$ for each $h \in \mathcal{A}_r$, since $D\lambda = 0$, in particular, for g_k such that $(\partial g_k(z,\tilde{z})/\partial z).1 = f_k(z)$ and satisfying 2.5.1.(1) by Lemma 2.5.1 on U, $k=1,2$, since $g(\ _1z,\ _2z)$ is $\ _1z$-superdifferentiable. Therefore, $\int_\gamma (f_1(z)\lambda_1 + f_2(z)\lambda_2)dz = (\int_\gamma f_1(z)dz)\lambda_1 + (\int_\gamma f_2(z)dz)\lambda_2$ for each constants λ_1 and $\lambda_2 \in \mathcal{A}_r$, consequently, $(\hat{f}_k(z).h)\lambda = (f_k(z)\lambda)\hat{}.h$ for each $h \in \mathcal{A}_r$. But this certainly does not mean that $(\int_\gamma f(z)dz)\lambda$ and $\lambda(\int_\gamma f(z)dz)$ are equal.

2.8. Remark. Let η be a differential form on open subset U of the Euclidean space $\mathbf{R^{2^r m}}$ with values in $\mathcal{A}_r$, then it can be written as

$$(1) \quad \eta = \sum_\Upsilon \eta_\Upsilon db^{\wedge \Upsilon},$$

where $b = (\ ^1b, ...,\ ^mb) \in \mathbf{R^{2^r m}}$, $\ ^jb = (\ ^jb_1, ...,\ ^jb_{2^r})$, $\ ^jb_k \in \mathbf{R}$, $\eta_\Upsilon = \eta_\Upsilon(b) : \mathbf{R^{2^r m}} \to \mathcal{A}_r$ are s times continuously differentiable $\mathcal{A}_r$-valued functions with $s \in \mathbf{N}$, $\Upsilon = (\Upsilon(1), ..., \Upsilon(m))$, $\Upsilon(j) = (\Upsilon(j,1), ..., \Upsilon(j,2^r)) \in \mathbf{N^{2^r}}$ for each j, $db^{\wedge\Upsilon} = d\ ^1b^{\wedge\Upsilon(1)} \wedge ... \wedge d\ ^mb^{\wedge\Upsilon(m)}$, $d\ ^jb^{\wedge\Upsilon(j)} = d\ ^jb_1^{\Upsilon(j,1)} \wedge ... \wedge d\ ^jb_{2^r}^{\Upsilon(j,2^r)}$, where $d\ ^jb_k^0 := 1$, $d\ ^jb_k^1 = d\ ^jb_k$, $d\ ^jb_k^v = 0$ for each $v > 1$. If $s \ge 1$, then there is defined an (external) differential

$$d\eta = \sum_{\Upsilon,(j,k)} (\partial \eta_\Upsilon / \partial\ ^jb_k)(-1)^{\alpha(j,k)} db^{\wedge(\Upsilon + e(j,k))},$$

where $e(j,k) = (0,...,0,1,0,...,0)$ with 1 on the $2^r(j-1)+k$-th place, $\alpha(j,k) = (\sum_{p=1}^{j-1}\sum_{v=1}^{2^r} \Upsilon(p,v)) + \sum_{v=1}^{k-1}\Upsilon(j,v)$. Now use the relations

(2) $\ ^jb_1 = (\ ^jz + (2^r-2)^{-1}\{-\ ^jz + \sum_{s\in\hat{b}} s(\ ^jzs^*)\})/2$ and

$\ ^jb_p = (i_p(2^r-2)^{-1}\{-\ ^jz + \sum_{s\in\hat{b}} s(\ ^jzs^*)\} -\ ^jzi_p)/2$ for each $i_p \in \hat{b}$. Then η

can be expressed in variables z. Consider basic elements $S=({}^1S,...,{}^mS)$ and their ordered product $S^{\to\Upsilon}:=(...({}^1S^{\to\Upsilon(1)}\ {}^2S^{\to\Upsilon(2)})...)\ {}^mS^{\to\Upsilon(m)}$, where ${}^jS=({}^jS_1,...,{}^jS_{2^r})=(1,i_1,i_2,...,i_{2^r-1})$, ${}^jS^{\to\Upsilon(j)}=(...(i_1^{\Upsilon(j,2)}i_2^{\Upsilon(j,3)})...)i_{2^p-1}^{\Upsilon(j,2^r)}$, $S^0=1$. Then Equation (1) can be rewritten in the form:

$$(3)\quad \eta=\sum_{\Upsilon}\xi_{\Upsilon}d(Sb)^{\wedge\Upsilon},$$

where $Sb=({}^1S_1\ {}^1b_1,...,{}^1S_{2^r}\ {}^1b_{2^r},...,{}^mS_1\ {}^mb_1,...,{}^mS_{2^r}\ {}^mb_{2^r})\in\mathcal{A}_r^{2^rm}$,
$d\ {}^jS_k\ {}^jb_k={}^jS_kd\ {}^jb_k$,
$d(Sb)^{\wedge\Upsilon}:=(...((d\ {}^1S\ {}^1b)^{\wedge\Upsilon(1)}\wedge(d\ {}^2S\ {}^2b)^{\wedge\Upsilon(2)})\wedge...)\wedge(d\ {}^mS\ {}^mb)^{\wedge\Upsilon(m)}$,
$(d\ {}^vS\ {}^vb)^{\wedge\Upsilon(v)}:=(...((d\ {}^vS_1\ {}^vb_1)^{\Upsilon(v,1)}\wedge(d\ {}^vS_2\ {}^vb_2)^{\Upsilon(v,2)})\wedge...)\wedge(d\ {}^vS_{2^r}\ {}^vb_{2^r})^{\Upsilon(v,2^r)}$,
$\xi_{\Upsilon}:=\eta_{\Upsilon}(S^{\to\Upsilon})^*$. Relative to the external product $d\ {}^jb_1$ anticommutes with others basic differential 1-forms ${}^jS_kd\ {}^jb_k$; for $k=2,...,2^r$ these 1-forms commute with each other relative to the external product. This means that the algebra of Cayley-Dickson algebra differential forms is graded relative to the external product.

From Equation (2) it follows, that

(4) $db_1=(dz+d(2^r-2)^{-1}\{-z+\sum_{s\in\hat{b}}s(zs^*)\})/2$,
$db_p=(di_p(2^r-2)^{-1}\{-z+\sum_{s\in\hat{b}}s(zs^*)\}-dzi_p)/2$ for each $i_p\in\hat{b}$. Therefore, the right side of Equation (3) can be rewritten with $d\ {}^jzi_p$, $di_p((s\ {}^jz)s^*)$ on the right side, where $i_p\in\{1,i_1,...,i_{2^r-1}\}$, $s\in\hat{b}$. Here we can use also $di_p\ {}^j\tilde{z}$ and $d((2^r-2)^{-1}\{-\ {}^j\tilde{z}+\sum_{s\in\hat{b}}s({}^j\tilde{z}s^*)\})i_p$ depending on the considered representation either z or $\tilde{z}$ or $(z,\tilde{z})$ of functions and differential forms. These 1-forms do neither commute nor anticommute, since they are not pure elements of the graded algebra. For example,
$d\ {}^jz\wedge d\ {}^jz=2\sum_{1\le v<k\le 2^r-1}i_vi_kd\ {}^jb_{v+1}\wedge d\ {}^jb_{k+1}$;
$(d\ {}^jz)^{\wedge p}=p!\sum_{1\le v(1)<...<v(p)\le 2^r-1}(...(i_{v(1)}i_{v(2)})...)i_{v(p)}d\ {}^jb_{v(1)+1}\wedge...\wedge d\ {}^jb_{v(p)+1}$
for each $3\le p\le 2^r$. On the other hand Equation (1) can be rewritten using the identities (2). This shows, that the exterior differentiation operator ${}_{\mathcal{A}_r}d$ for $\mathcal{A}_r$-valued differential forms over $\mathcal{A}_r$ and that of for their real realization ${}_{\mathbf{R}}d$ coincide and their common operator is denoted by d. Consider the equality

$$(\partial\eta_{\Upsilon}/\partial\ {}^jb^l)\ {}^jb^l\wedge db^{\Upsilon}=[(\partial\eta_{\Upsilon}/\partial\ {}^jz).(\partial\ {}^jz/\partial\ {}^jb^l)]\ {}^jb^l\wedge db^{\Upsilon}$$
$$+[(\partial\eta_{\Upsilon}/\partial\ {}^j\tilde{z}).(\partial\ {}^j\tilde{z}/\partial\ {}^jb_l)]\ {}^jb^l\wedge db^{\Upsilon}.$$

Applying it to $l = 1, ..., 2^r$ and summing left anf right parts of these equalities we get $d\eta(z, \tilde{z}) = ((\partial\eta/\partial z).d\, {}^j z) \wedge db^{\Upsilon} + ((\partial\eta/\partial\tilde{z}).d\, {}^j \tilde{z}) \wedge db^{\Upsilon}$, hence the external differentiation can be presented in the form

$$(5) \quad d = \partial_z + \partial_{\tilde{z}},$$

where ∂_z and $\partial_{\tilde{z}}$ are external differentiations by variables z and $\tilde{z}$ respectively.

Certainly for an external product $\eta_1 \wedge \eta_2$ there is not (in general) a $\lambda \in \mathcal{A}_r$ such that $\lambda\eta_2 \wedge \eta_1 = \eta_1 \wedge \eta_2$, if η_1 and η_2 are not pure elements (even or odd) of the graded algebra of differential forms over $\mathcal{A}_r$.

2.9. Definition. A Hausdorff topological space X is said to be n-connected for $n \ge 0$ if each continuous map $f : S^k \to X$ from the k-dimensional real unit sphere into X has a continuous extension over $\mathbf{R^{k+1}}$ for each $k \le n$. A 1-connected space is also said to be simply connected.

2.10. Remark. In accordance with Theorem 1.6.7 [31] a space X is n-connected if and only if it is path connected and $\pi_k(X, x)$ is trivial for every base point $x \in X$ and each k such that $1 \le k \le n$.

Denote by $Int(U)$ an interior of a subset U in a topological space X, by $cl(U) = \bar{U}$ a closure of U in X. For a subset U in $\mathcal{A}_r$, let $\pi_{s,p,t}(U) := \{u : z \in U, z = \sum_{v \in \mathbf{b}} w_v v,\ u = w_s s + w_p p\}$ for each $s \ne p \in \mathbf{b}$, where $t := \sum_{v \in \mathbf{b} \setminus \{s,p\}} w_v v \in \mathcal{A}_{r,s,p} := \{z \in \mathcal{A}_r : z = \sum_{v \in \mathbf{b}} w_v v,\ w_s = w_p = 0,\ w_v \in \mathbf{R}\ \forall v \in \mathbf{b}\}$. That is, geometrically $\pi_{s,p,t}(U)$ is the projections on the complex plane $\mathbf{C}_{s,p}$ of the intersection of U with the plane $\tilde{\pi}_{s,p,t} \ni t$, $\mathbf{C}_{s,p} := \{as + bp : a, b \in \mathbf{R}\}$, since $sp^* \in \hat{b}$.

2.11. Theorem. *Let U be a domain in $\mathcal{A}_r$, $r \ge 3$, such that $\emptyset \ne Int(U) \subset U \subset cl(Int(U))$ and U is $(2^r - 1)$-connected; $\pi_{s,p,t}(U)$ is simply connected in $\mathbf{C}$ for each $k = 0, 1, ..., 2^{r-1} - 1$, $s := i_{2k}$, $p := i_{2k+1}$, $t \in \mathcal{A}_{r,s,p}$ and $u \in \mathbf{C}_{s,p}$ for which there exists $z = u + t \in U$ (see §2.10). Suppose $f \in C^0_b(U, \mathcal{A}_r)$ and f is superdifferentiable by $z \in U$ and f has a continuous extension on an open domain W such that $W \supset U$. Then for each rectifiable closed path γ in U a Cayley-Dickson line integral $\int_\gamma f(z)dz = 0$ is equal to zero.*

Proof. In view of Proposition 2.3 f is z-represented and $\partial_{\tilde{z}} f = 0$ on U. Therefore, $\xi({}_1z, {}_2z)$ is independent from ${}_2z$, where ξ is the corresponding to f function from §2.5, consequently, $g({}_1z, {}_2z)$ is also independent from ${}_2z$ and we can write $g(z)$ shortly. For a path γ there exists a compact canonical closed subset in $\mathcal{A}_r$: $W \subset Int(U)$ such that $\gamma([0, 1]) \subset W$,

since γ is rectifiable and $\mathcal{A}_r$ is locally compact. In view of Theorem 2.7 for each sequence of functions $f_n \in C^1(U, \mathcal{A}_r)$ converging to f in $C^0_b(U, \mathcal{A}_r)$ such that $f_n(z) = (\partial g_n(z)/\partial z).1$ with $g_n(z) \in C^2(U, \mathcal{A}_r)$ and satisfying conditions of §2.5, since ξ is independent from ${}_2z$, and each sequence of paths $\gamma_n : [0,1] \to U$ C^3-continuously differentiable and converging to γ relative to the total variation $V(\gamma - \gamma_n)$ there exists $\lim_n \int_{\gamma_n} f_n(z, \tilde{z})dz = \int_\gamma f(z)dz$. Therefore, it is sufficient to consider the case of $f \in C^1(U, \mathcal{A}_r)$ such that $f(z) = (\partial g(z)/\partial z).1$ on U, and continuously differentiable γ, where $g \in C^2(U, \mathcal{A}_r)$ satisfies conditions of §2.5. Denote the integral $\int_\gamma f(z)dz$ by Q. We can write this integral in the form $Q = \int_0^1 (\partial g(z)/\partial z).\gamma'(t)dt$. Write f in the form:

$f = \sum_{s\in \mathbf{b}} f_s s = \sum_{\beta=0}^{2^{r-1}-1} f_{2\beta,2\beta+1}$, where $f_{\beta,\nu} := f_{i_\beta} i_\beta + f_{i_\nu} i_\nu$, $f_s \in \mathbf{R}$ for each $s \in \mathbf{b}$. Therefore,

$d\gamma(t) = \gamma'(t)dt = \sum_{j=0}^{2^{r-1}-1} {\gamma'}_{2j,2j+1}(t)dt$. The condition $\gamma(0) = \gamma(1)$ is equivalent to $\gamma_{2j,2j+1}(0) = \gamma_{2j,2j+1}(1)$ for each $j = 0, 1, ..., 2^{r-1} - 1$. We have $\gamma_{\beta,\nu} \subset \pi_{\beta,\nu,t}(U)$ for each $\beta \ne \nu \in \mathbf{b}$. The multiplication in $\mathcal{A}_r$ is distrbutive, consequently,

$(\partial g(z)/\partial z).(d\gamma(t)) = \sum_{k=0}^{2^{r-1}-1} (\partial g(z)/\partial z).(d\gamma_{2k,2k+1}(t))$.

In view of the Hurewicz isomorphism theorem (see §7.5.4 [31]) $H_q(U, x) = 0$ for each $x \in U$ and each $q < 2^r$, hence $H^l(U, x) = 0$ for each $l \ge 1$.

If $f : Y \to V$ is continuous, then $r \circ f : Y \to \Omega$ is continuous, if f is onto V, then $r \circ f$ is onto Ω, where $r : V \to \Omega$ is a retraction, V, Y and Ω are topological spaces. The topological space U is metrizable, hence for each closed subset Ω in U there exists a canonical closed subset $V \subset U$ such that $V \supset \Omega$ and Ω is a retraction of V, that is, there exists a continuous mapping $r : V \to \Omega$, $r(z) = z$ for each $z \in \Omega$ (see [9] and Theorem 7.1 [16]). Therefore, if V is a $(2^r - 1)$-connected canonical closed subset of U and Ω is a two dimensional C^0-manifold such that Ω is a retraction of V, then Ω is simply connected, since each continuous mapping $f : S^k \to \Omega$ with $k \le 1$ has a continuous extension $f : \mathbf{R^{k+1}} \to V$ and $r \circ f : \mathbf{R^{k+1}} \to \Omega$ is also a continuous extension of f from S^k on $\mathbf{R^{k+1}}$.

From (2^r-1)-connectedness of U it follows, that there is a two dimensional real differentiable manifold Ω contained in U such that $\partial\Omega = \gamma$. This may be lightly seen by considering partitions Z_n of U by $S^n_{l,k} \cap U$ and taking $n \to \infty$, where $S^n_{l,k}$ are parallelepipeds in $\mathcal{A}_r$ with ribs of length n^{-1}, l, k and $n \in \mathbf{N}$, two dimensional faces ${}_1S^n_l$ and 2^{r-1}-dimensional faces ${}_2S^n_k$ of

$S^n_{l,k} = {}_1S^n_l \times {}_2S^n_k$ are parallel to $\mathbf{C}_{s,p}$ or $\mathcal{A}_{r,s,p}$ with $s = i_{2k}$ and $p = i_{2k+1}$ respectively such that there exists a sequence of paths γ_n converging to γ relative to $|*|_{\mathcal{A}_r}$ and a sequence of (continuous) two dimensional C^0-manifolds Ω^n with $\partial\Omega^n = \gamma^n$, $\Omega^n \subset \bigcup_{l,k}[(\partial\ {}_1S^n_l) \times (\partial\ {}_2S^n_k)]$. Choose Ω orientable and of class C^3 as Riemann manifolds such that taking their projections on $\mathbf{C}_{s,p}$ the corresponding paths $\gamma_{2k,2k+1}$ and regions $\Omega_{s,p}$ in $\mathbf{C}_{s,p}$ satisfy the conditions mentioned above in this proof.

To the appearing integrals the classical (generalized) Stokes theorem can be applied (see Theorem V.1.1 [36]):

$\int_{\Omega_{2k,2k+1}} \eta(v) = \int_0^1 (\partial g(z)/\partial z).\gamma'_{2k,2k+1}(t)dt$,

where $\Omega_{2k,2k+1}$ is the simply connected domain in $\mathbf{C}_{i_{2k},i_{2k+1}}$ such that $\partial\Omega_{2k,2k+1} = \gamma_{2k,2k+1}$ for each k, $\eta(v) = d[(\partial g(z)/\partial z).dv]$, $v = z_{2k,2k+1} \in \Omega_{2k,2k+1} \subset \mathbf{C}_{i_{2k},i_{2k+1}}$. The function g is in $C^2(U, \mathcal{A}_r)$, hence $(D^2g(z)).(h_1, h_2) := (D^2g(z)).(h_2, h_1)$ for each h_1 and h_2 in $\mathcal{A}_r$. Therefore, due to conditions of §2.5 imposed on g we have

$\int_{\Omega_{2k,2k+1}} \eta(v) = \int_{\Omega_{2k,2k+1}} d[(\partial g(z)/\partial z).dv] = \int_{\Omega_{2k,2k+1}} d^2q(z_{2k,2k+1}) = 0$

for each $k = 0, 1, ..., 2^{r-1} - 1$, since

$(\partial g(z)/\partial z).\gamma'_{2k,2k+1} = (\partial g(z)/\partial z).(s(\gamma'_{2k} + \gamma'_{2k+1}s^*p))$, such that

$(\partial g(z)/\partial z).\gamma'_{2k,2k+1} = (\partial g(z)/\partial w_s).\gamma'_{2k} + (\partial g(z)/\partial w_p).\gamma'_{2k+1}$

$= (\partial g(z)/\partial z_{2k,2k+1}).(\gamma'_{2k} + \gamma'_{2k+1}s^*p)$,

$\partial y(z)/\partial w_s = (\partial y(z)/\partial z_{2k,2k+1} + \partial y(z)/\partial \bar{z}_{2k,2k+1}).1$ and

$\partial y(z)/\partial w_p = (\partial y(z)/\partial z_{2k,2k+1} - \partial y(z)/\partial \bar{z}_{2k,2k+1}).(s^*p)$ for each z-superdifferentiable function y on U and $\partial g(z)/\partial \bar{z}_{2k,2k+1} = 0$, where q corresponds to $g|_{\Omega_{2k,2k+1}}$, $s = i_{2k}$, $p = i_{2k+1}$.

2.12. Definitions. A continuous function on an open domain U in $\mathcal{A}_r$ such that $\emptyset \ne U$ and $\int_\gamma f dz = 0$ for each rectifiable closed path γ in U, then f is called $\mathcal{A}_r$-integral holomorphic on U (see §2.5).

If f is a z-superdifferentiable function on U, then it is called $\mathcal{A}_r$-holomorphic on U.

2.13. Corollary. *Let f be $\mathcal{A}_r$-holomorphic function on an open $(2^r - 1)$-connected domain U in $\mathcal{A}_r$ such that $\pi_{s,p,t}(U)$ is simply connected in $\mathbf{C}_{s,p}$ for each $t \in \mathcal{A}_{r,s,p}$ and $u \in \mathbf{C}_{s,p}$, $s := i_{2k}$, $p := i_{2k+1}$ for which there exists $z = t + u \in U$, then f is $\mathcal{A}_r$-integral holomorphic.*

This follows immediately from Theorem 2.11.

2.14. Definition. Let U be a subset of $\mathcal{A}_r$ and $\gamma_0 : [0, 1] \to \mathcal{A}_r$ and $\gamma_1 : [0, 1] \to \mathcal{A}_r$ be two continuous paths. Then γ_0 and γ_1 are called homotopic

relative to U, if there exists a continuous mapping $\gamma : [0,1]^2 \to U$ such that $\gamma([0,1],[0,1]) \subset U$ and $\gamma(t,0) = \gamma_0(t)$ and $\gamma(t,1) = \gamma_1(t)$ for each $t \in [0,1]$.

2.15. Theorem. *Let W be an open subset in $\mathcal{A}_r$, $r \ge 3$, and f be an $\mathcal{A}_r$-holomorphic function on W with values in $\mathcal{A}_r$. Suppose that there are two rectifiable paths γ_0 and γ_1 in W with common initial and final points ($\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$) homotopic relative to U, where U is a $(2^r - 1)$-connected subset in W such that $\pi_{s,p,t}(U)$ is simply connected in $\mathbf{C}$ for each $t \in \mathcal{A}_{r,s,p}$ and $u \in \mathbf{C}_{s,p}$, $s = i_{2k}$, $p = i_{2k+1}$, $k = 0, 1, ..., 2^{r-1} - 1$, for which there exists $z = u + t \in U$. Then $\int_{\gamma_0} f dz = \int_{\gamma_1} f dz$.*

Proof. A homotopy of γ_0 with γ_1 realtive to U implies homotopies of $(\gamma_0)_{2j,2j+1}$ with $(\gamma_1)_{2j,2j+1}$ relative to $\pi_{2j,2j+1,t}(U)$ in $\mathbf{C}_{s,p}$ with $s = i_{2j}$ and $p = i_{2j+1}$ for each $j = 0, 1, 2^{r-1} - 1$ for each $t \in \mathcal{A}_{r,s,p}$ and $u \in \mathbf{C}_{s,p}$ for which there exists $z = t + u \in U$. Consider a path ζ such that $\zeta(t) = \gamma_0(2t)$ for each $0 \le t \le 1/2$ and $\zeta(t) = \gamma_1(2 - 2t)$ for each $1/2 \le t \le 1$. Then ζ is a closed path contained in a U. In view of Theorem 2.11 $\int_\zeta f(z)dz = 0$. On the other hand, $\int_\zeta f(z)dz = \int_{\gamma_0} f(z)dz - \int_{\gamma_1} f(z)dz$, consequently, $\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$.

2.15.1. Corollary. *Let $f \in C^1$ satisfies conditions of Theorem 2.15. Then for each $z \in U$ there exists $(\partial(\int_\gamma f(\zeta)d\zeta)/\partial z).h = \hat{f}(z).h$ for each $h \in \mathcal{A}_r$, where $\gamma(0) = z_0$, $\gamma(1) = z$, z_0 is a marked point in U.*

2.16. Theorem. *Let f be a $\mathcal{A}_r$ locally z-analytic function on an open domain U in $\mathcal{A}_r^n$, then f is $\mathcal{A}_r$-holomorphic on U.*

Proof. From the definition of the superdifferential we get $(Dz^n).h = \sum_{k=0}^{n-1} z^k h z^{n-k-1}$. Using the formula of the superdifferential for a product of functions, from §2.7 we obtain, that each f of the form $(2.5.i, ii)$ is superdifferentiable (by z) when $n_0 \ge 0$ in $(2.5.ii)$. Using the norm $\| * \|_\omega$-convergence of series with respect to z for a given $f \in C^\omega(U, \mathcal{A}_r)$ we obtain for each $a \in U$, that there exists its neighbourhood W, where f is $\mathcal{A}_r$-holomorphic, hence f is $\mathcal{A}_r$-holomorphic on U.

2.17. Note. In the next section it is shown that an octonion-holomorphic function is infinite differentiable; furthermore, under suitable conditions equivalences between the properties of octonion holomorphicity, octonion integral holomorphicity and octonion local z-analyticity, will be proved there too. Integral (2.5.4) may be generalized for a continuous function $q : U \to \mathcal{A}_r$ such that $V(q \circ \gamma) < \infty$. Substituting Δz_k on $q(z_{k+1}) - q(z_k) =: \Delta q_k$ in Formula

(2.5.5) we get

$$(1) \quad \int_\gamma f(z,\tilde{z})dq(z) := \lim_P I(f, q\circ\gamma; P), \text{ where}$$

$$(2) \quad I(f, q\circ\gamma; P) = \sum_{k=0}^{q-1} \hat{f}(z_{k+1}, \tilde{z}_{k+1}).(\Delta q_k).$$

In paticular, if $\gamma \in C^1$ and q is $\mathcal{A}_r$-holomorphic on U, also $f(z,\tilde{z}) = (\partial g(z,\tilde{z})/\partial z).1$, where $g \in C^1(U, \mathcal{A}_r)$, then

$$\int_\gamma f(z,\tilde{z})dq(z) = \int_0^1 (\partial g(z,\tilde{z})/\partial z).((\partial_z q(z)|_{z=\gamma(s)}).\gamma'(s))ds$$

and $V(\gamma) \le \int_0^1 |\gamma'(s)|ds$.

Let $f : U \to \mathcal{A}_r$ be an $\mathcal{A}_r$-holomorphic function on U, where U is an open subset of $\mathcal{A}_r^n$. If there exists an $\mathcal{A}_r$-holomorphic function $g : U \to \mathcal{A}_r$ such that $g'(z).1 = f(z)$ for each $z \in U$, then g is called a primitive of f.

2.18. Proposition. *Let U be an open connected subset of $\mathcal{A}_m^n$, $m \ge 3$, and g be a primitive of f on U, then a set of all primitives of f is: $\{h : h = g + C, C = const \in \mathcal{A}_m\}$.*

Proof. In view of Lemma 2.5.1 for each two primitives g_1 and g_2 of f for each $z \in U$ there exists a ball $B \subset U$, $z \in U$, such that $(g_1 - g_2)|_U = const \in \mathcal{A}_m$ (see §2.7). Suppose $h'(z) = 0$ for each $z \in U$, then consider $q(s) := h((1-s)a + sz)$ for each $s \in [0,r]$, where a is a marked point in U and $B(a, r, \mathcal{A}_m)$ is a ball contained in U, $r > 0$, $z \in B(a, r, \mathcal{A}_m)$. Then q is correctly defined and $q(0) = q(1)$. Therefore, the set $V := \{z \in U : h(z) = h(a)\}$ is open in U, since with each point a it contains its neighbourhood. On the other hand, it is closed due continuity of h, hence $V = U$, since U is connected, consequently, $h = const$ on U.

3 Meromorphic functions and their residues.

At first we define and describe the exponential and the logarithmic functions of octonion variables and then apply them to the investigation of octonionic residues. Moreover, these studies are accomplished also for variables in Cayley-Dickson algebras $\mathcal{A}_r$ for each $r \ge 4$.

3.1. Note and Definition. For a variable $z \in \mathcal{A}_r$ with $r \geq 3$ put

$$(3.1.) \quad \exp(z) := \sum_{n=0}^{\infty} z^n/n!.$$

In view of Note 2.1 z^n and hence $\exp(z)$ are correctly defined, since real numbers commute with each element of $\mathcal{A}_r$, $n! \in \mathbf{N} \subset \mathbf{R}$. If $|z| \leq R < \infty$, then the series (3.1) converges, since $|exp(z)| \leq \sum_{n=0}^{\infty} |z^n/n!| \leq \exp(R) < \infty$. Therefore, $\exp : \mathcal{A}_r \to \mathcal{A}_r$. The restriction of exp on each of the subsets $\mathbf{Q}_s := \{z : z \in \mathcal{A}_r, z = a + bs,\ a, b \in \mathbf{R}\}$ is commutative, where $s \in \hat{b}$, $\hat{b}_r := \mathbf{b}_r \setminus \{1\}$, $\mathbf{b} := \mathbf{b}_r$, $\hat{b} := \hat{b}_r$, but in general two elements z_1 and $z_2 \in \mathcal{A}_r$ do not commute and the function $\exp(z_1 + z_2)$ on $\mathcal{A}_r^2$ does not coincide with $\exp(z_1)\exp(z_2)$.

3.2. Proposition. *Let $z \in \mathcal{A}_r$, $r \geq 3$, be written in the form $z = v + M$, where $v \in \mathbf{R}$, $M \in \mathcal{I}_r$, $\mathcal{I}_r := \{\eta \in \mathcal{A}_r : Re(\eta) = 0\}$, then*

$$(3.2) \quad \exp(z) = \exp(v)\exp(M), \ \ \textit{where}$$

$$(3.3) \quad \exp(M) = (\cos|M|) + [(\sin|M|)/|M|]M$$

for $M \neq 0$ and $\exp(0) = 1$.

Proof. Consider $M \in \mathcal{I}_r$ and write it in the form $M = a + bl$, where $l = i_{2^{r-1}}$ is the element of the doubling procedure of $\mathcal{A}_r$ from $\mathcal{A}_{r-1}$, $a, b \in \mathcal{A}_{r-1}$, $Re(a) = 0$. Then
$M^2 = a^2 + a(bl) + a^*(bl) - (bl)(bl)^* = -(|a|^2 + |b|^2)$,
since $a^* = -a$, $(bl)(bl)^* = (bl)^*(bl) = (lb^*)(bl^{-1}) = b^*b = bb^*$, consequently, $M^{2n} = (-|M|^2)^n$, $M^{2n+1} = (-|M|^2)^n M$ for each $1 \leq n \in \mathbf{Z}$. Therefore,

$$\exp(M) = 1 + \sum_{n=1}^{\infty}(-|M|^2)^n/(2n)! + \sum_{n=0}^{\infty}(-|M|^2)^n M/(2n+1)!$$

$$= (\cos|M|) + [(\sin|M|)/|M|]M$$

for each $M \neq 0$, $\exp(0) = 1$. Since $\lim_{0 \neq \phi \to 0} \sin(\phi)/\phi = 1$, where $\phi \in \mathbf{R}$, then the limit taken in Formula (3.3) while $|M| \neq 0$ tends to 0 gives the particular case $\exp(0) = 1$. Since $v \in \mathbf{R}$ commutes with M, $[v, M] = 0$, then $\exp(v + M) = \exp(v)\exp(M)$.

3.3. Corollary. *If $z \in \mathcal{A}_r$, $r \geq 3$, is written in the form $z = \sum_{s \in \mathbf{b}} w_s s$ with real w_s for each $s \in \mathbf{b}$, then $|\exp(z)| = \exp(v)$.*

Proof. If $\sum_{s\ne 1} w_s^2 = 0$ this is evident. Suppose $\sum_{s\ne 1} w_s^2 \ne 0$. In view of Formulas $(3.2, 3.3)$

$$(3.4)\quad exp(z) = exp(v)A, \text{ where } A = \cos|M| + [(\sin|M|)/|M|]M$$

Since $A \in \mathcal{A}_r$, then $A^* = \cos|M| - [(\sin|M|)/|M|]M$, $M^* = -M$ and inevitably $|\exp(z)| = \exp(v)$.

3.4. Corollary. *The function* $\exp(z)$ *on the set* $\mathcal{I}_r := \{z : z \in \mathcal{A}_r, Re(z) = 0\}$ *is periodic with* $(2^r - 1)$ *generators of periods* $s \in \hat{b}_r$ *such that* $\exp(z(1 + 2\pi n/|z|)) = \exp(z)$ *for each* $0 \ne z \in \mathcal{I}_r$ *and each integer number* n*. If* $z \in \mathcal{A}_r$ *is written in the form* $z = 2\pi sM$*, where* $M \in \mathcal{I}_r$*,* $|M| = 1$*, then* $\exp(z) = 1$ *if and only if* $s \in \mathbf{Z}$*.*

Proof. In view of Formulas $(3.2, 3.3)$ $\exp(sM) = 1$ for a given $z = sM \in \mathcal{I}_r$ with $|M| = 1$ if and only if $\cos(s|M|) = 1$ and $\sin(s|M|) = 0$, that is equivalent to $s \in \{2\pi n : n \in \mathbf{Z}\}$, since $|M| = 1$ by the hypothesis of this corollary. The particular cases of Formula (3.3) are: $w_{s_0} \ne 0$ and $w_s = 0$ for each $s \ne s_0$ in $\hat{b} := \hat{b}_r$, hence $s \in \hat{b}$ are $(2^r - 1)$ generators for the periods of exp.

3.5. Corollary. *The function* exp *is the epimorphism from* $\mathcal{I}_r$ *on the* $(2^r - 1)$*-dimensional unit sphere* $S^{2^r-1}(0, 1, \mathcal{A}_r) := \{z : z \in \mathcal{A}_r, |z| = 1\}$*.*

Proof. In view of Corollary 3.3 the image $\exp(\mathcal{I}_r)$ is contained in $S^{2^r-1}(0, 1, \mathcal{A}_r)$. The sphere $S^{2^r-1}(0, 1, \mathcal{A}_r)$ is characterized by the condition $\sum_{s\in\mathbf{b}} w_s^2 = 1$ or $w_1^2 + |M_1|^2 = 1$, where $M_1 \in \mathcal{I}_r$. To show that $\exp(\mathcal{I}_r) = S^{2^r-1}(0, 1, \mathcal{A}_r)$ it is sufficient to find $z = v + M$, where $v \in \mathbf{R}$, $M \in \mathcal{I}_r$, such that $w_1 = \cos|M|$, $M_1 = [(\sin|M|)/|M|]M$. For this take $|M| = \arccos w_1 \in [0, \pi]$, since $w_1 \in [-1, 1]$ and if $|w_1| \ne 1$ put $M = M_1(1 - w_1^2)^{-1/2} \arccos w_1$. In particular, for $w_1 = 1$ take $M = 0$; for $w_1 = -1$ take $M = \pi q$, where $q \in \hat{b}$.

3.6. Corollary. *Each element of the Cayley-Dickson algebra* $\mathcal{A}_r$*,* $r \ge 3$*, has a polar decomposition*

$$(3.5)\quad z = \rho \exp(2\pi(\sum_{s\in\hat{b}} \phi_s s)),$$

where $\phi_s \in [-1, 1]$ *for each* $s \in \hat{b}$*,* $\sum_{s\in\hat{b}} \phi_s^2 = 1$*,* $\rho := |z|$*.*

Proof. This follows from Formulas $(3.2, 3.3)$ and Corollary 3.5.

3.6.1. Definition. Let $\mathcal{A}_\infty$ denotes the family consisting of elements $z = \sum_{s\in\mathbf{b}} w_s s$ such that $\tilde{z} := w_1 - \sum_{s\in\hat{b}} w_s s$, $z\tilde{z} =: |z|^2 = \sum_{s\in\mathbf{b}} w_s^2 < \infty$,

where $\mathbf{b} := \mathbf{b}_\infty := \bigcup_{r=2}^\infty \mathbf{b}_r = \{1, i_1, i_2, ..., i_{2^r}, ...\}$, $\hat{b} := \mathbf{b} \setminus \{1\}$, $w_s \in \mathbf{R}$ for each s.

3.6.2. Theorem. *The family $\mathcal{A}_\infty$ has the structure of the normed power-associative left and right distributive algebra over $\mathbf{R}$ with the external involution of order two.*

Proof. Let $\mathcal{I}_\infty := \{z \in \mathcal{A}_\infty : Re(z) = 0\}$. Then each $M \in \mathcal{I}_\infty$ is the limit of the sequence $M_r \in \mathcal{I}_r$, also $|z| =: \rho$ is the limit of the sequence $\rho_r := |z_r|$, where $z_r \in \mathcal{A}_r$. Therefore, $z = \lim_{r\to\infty} z_r = \lim_{r\to\infty} \rho_r\{\cos|M_r| + [(\sin|M_r|)/|M_r|]M_r\} = \rho\{\cos|M| + [(\sin|M|)/|M|]M\} = \rho\exp(M)$. Using the polar coordinates (ρ, M) prove the power-associativity. There exists the natural projection P_r from $\mathcal{A}_\infty$ onto $\mathcal{A}_r$ for each $r \ge 2$ given by the formulas: $M_r := \{\sum_{s\in\hat{b}_r} m_s s\}|M|[\sum_{s\in\hat{b}_r} m_s^2]^{-1/2}$ for $\sum_{s\in\hat{b}_r} m_s^2 \ne 0$ and $M_r = 0$ in the contrary case for each $M = \sum_{s\in\hat{b}} m_s s \in \mathcal{I}_\infty$, where $m_s \in \mathbf{R}$ for each $s \in \hat{b}$; then $z_r := P_r(z) = \rho_r\{\cos|M_r| + [(\sin|M_r|)/|M_r|]M_r\}$, where $\rho_r = (\sum_{s\in\mathbf{b}_r} w_s^2)^{1/2}$, $z = \sum_{s\in\mathbf{b}} w_s s$, $\lim_{0\ne\phi\to 0}\sin(\phi)/\phi = 1$. Then $\lim_{r\to\infty} z_r = z$ relative to the norm $|z|$ in $\mathcal{A}_\infty$. Therefore, for each $n \in \mathbf{Z}$ there exists $\lim_{r\to\infty}(\rho_r)^n \exp(nM_r) = \rho^n \exp(nM) = z^n$, consequently, $\mathcal{A}_\infty$ is power-associative, since each $\mathcal{A}_r$ is power-associative, cos and sin are continuous functions. Evidently, $\mathcal{A}_\infty$ is the $\mathbf{R}$-linear space. The continuity of multiplication relative to the norm $|z|$ follows from the inequalities $|\xi_r\eta_r - \psi_r\zeta_r| \le |\xi_r\eta_r - \xi_r\zeta_r| + |\xi_r\zeta_r - \psi_r\zeta_r| \le |\xi||\eta-\zeta| + |\xi-\psi||\zeta|$ and taking the limit with r tending to the infinity, since $|\xi_r| \le |\xi|$ and $|\xi_r\eta_r| \le |\xi_r||\eta_r|$ for each $\xi_r, \eta_r \in \mathcal{A}_r$ and for each $r \in \mathbf{N}$. Left and right distributivity $(\xi + \psi)\zeta = \xi\zeta + \psi\zeta$ and $\zeta(\xi + \psi) = \zeta\xi + \zeta\psi$ follow from taking the limit with r tending to the infinity and such distributivity in each $\mathcal{A}_r$. The involution $z \mapsto \tilde{z} =: z^*$ in $\mathcal{A}_r$ is of order two, since $(z^*)^* = z$. It is external, since there is not any finite algebraic relation with constants in $\mathcal{A}_\infty$ transforming the variable $z \in \mathcal{A}_\infty$ into z^*. The relation $z^* = \lim_{r\to\infty}(2^r - 2)^{-1}\{-z_r + \sum_{s\in\hat{b}_r} s(z_r s^*)\} = \lim_{r\to\infty}(2^r - 2)^{-1}\{-z + \sum_{s\in\hat{b}_r} s(zs^*)\}$ is of infinite order. The relations of the type $z_r^* = l_{r+1}(z_r l_{r+1}^*)$ in $\mathcal{A}_r$ use external automorphism with $l_{r+1} := i_{2^r} \in \mathcal{A}_{r+1} \setminus \mathcal{A}_r$, moreover, the latter relation is untrue for z^* and $z \in \mathcal{A}_\infty$ instead of $z_r \in \mathcal{A}_r$.

No any finite set of non-zero constants $a_1, ..., a_n \in \mathcal{A}_\infty$ can provide the automorphism $z \mapsto \tilde{z}$ of $\mathcal{A}_\infty$. To prove this consider an $\mathbf{R}$-subalgebra $\Upsilon_{M_1,...,M_n}$ of $\mathcal{A}_\infty$ generated by $\{M_1, ..., M_n\}$, where $a_j = |a_j|e^{M_j}$, $M_j \in \mathcal{I}_\infty$. Since $a_1a_1^* = |a_1|^2 > 0$, then $\mathbf{R}|a_1|^2 = \mathbf{R} \subset \Upsilon_{M_1,...,M_n}$, hence $1 \in \Upsilon_{M_1,...,M_n}$. If

$\Upsilon_{M_1,...,M_n} = \mathbf{R}$, then it certainly can not provide the automorphism $z \mapsto \tilde{z}$ of $\mathcal{A}_\infty$. Consider $\Upsilon_{M_1,...,M_n} \ne \mathbf{R}$, without loss of generality suppose $a_1 \notin \mathbf{R}$. There is the scalar product $Re(z\tilde{y})$ in $\mathcal{A}_\infty$ for each $z, y \in \mathcal{A}_\infty$. Let b_1 be the projection of a_1 in a subspace of $\mathcal{A}_\infty$ orthogonal to $\mathbf{R}1$, then by our supposition $b_1 \ne 0$ and $b_1 \in \mathcal{I}_\infty$. Therefore, $b_1^2/|b_1|^2 = -1$, consequently, Υ_{M_1} is isomorphic to $\mathbf{C}$. Certainly, no any $\mathcal{A}_r$, $r \in \mathbf{N}$, can provide the automorphism $z \mapsto \tilde{z}$ of $\mathcal{A}_\infty$. Therefore, without loss of generality suppose, that $\Upsilon_{M_1,...,M_n}$ is not isomorphic to $\mathbf{C}$ and $a_2 \notin \mathbf{C}$. If $M, N \in \mathcal{I}_r$ and $Re(MN^*) = 0$, then $MN \in \mathcal{I}_r$ and hence $(MN)^* = NM = -MN$, since $A^* = -A$ for each $A \in \mathcal{I}_r$. Let b_2 be the projection of M_2 in a susbspace of $\mathcal{A}_\infty$ orthogonal to Υ_{M_1} relative to the scalar product $Re(z\tilde{y})$. Then $b_2 \ne 0$ by our supposition and $b_2 \in \mathcal{I}_\infty$, $b_2^2/|b_2|^2 = -1$, hence after the doubling procedure with $b_2/|b_2|$ we get, that Υ_{M_1,M_2} is a subalgebra of $\mathcal{A}_4$. Then proceed by induction, suppose $\Upsilon_{M_1,...,M_k}$ is a subalgebra of $\mathcal{A}_{2^k}$, $k \in \mathbf{N}$, $k < n$. Since $\mathcal{A}_{2^k}$ can not provide the automorphism $z \mapsto \tilde{z}$ of $\mathcal{A}_\infty$, suppose without loss of generality, that $a_{k+1} \notin \Upsilon_{M_1,...,M_k}$ and consider the orthogonal projection b_{k+1} of M_{k+1} in a subspace of $\mathcal{A}_\infty$ orthogonal to $\Upsilon_{M_1,...,M_k}$ relative to the scalar product $Re(z\tilde{y})$. Then $b_{k+1} \ne 0$ and $b_{k+1} \in \mathcal{I}_\infty$, $b_{k+1}^2/|b_{k+1}|^2 = -1$. Then the doubling procedure with $b_{k+1}/|b_{k+1}|$ gives the algebra $\Upsilon_{M_1,...,M_{k+1}}$ which is the subalgebra of $\mathcal{A}_{2^{k+1}}$, etc. As the result $\Upsilon_{M_1,...,M_n}$ is the subalgebra of $\mathcal{A}_{2^n}$ and can not provide the automorphism $z \mapsto \tilde{z}$ of $\mathcal{A}_\infty$, where $a_1, ..., a_n \in \Upsilon_{M_1,...,M_n}$ due to the formula of polar decomposition (see (3.2,3)) of Cayley-Dickson numbers.

3.6.3. Note. From Theorem 3.6.2 it follows, that $\mathcal{A}_\infty$ together with $\mathbf{C} = \mathcal{A}_1$ are two extreme cases, where the conjugation $z \to z^*$ is the external automorphism, though the complex field $\mathbf{C}$ is easier to work due to its commutativity and associativity, than $\mathcal{A}_\infty$ which is neither commutative nor associative. In view of Definition 3.6.1 and Theorem 3.6.2 preceding results can be transferred from the case $\mathcal{A}_r$ with $r \ge 4$ on $\mathcal{A}_\infty$. Definitions $2.1 - 2.2$ are transferrable on $\mathcal{A}_\infty$, moreover, in view of algebraic independence of z and z^* in $\mathcal{A}_\infty$ we have that z and z^* are automatically independent variables and having projections $P_r : \mathcal{A}_\infty \to \mathcal{A}_r$ for each $r \ge 2$ justify our definitions of $C^\omega_{z,\tilde{z}}(U, \mathcal{A}_r)$ and $C^n_{z,\tilde{z}}(U, \mathcal{A}_r)$, also it justifies axioms of superdifferentiations in §2.2. Evidently Propositions 2.2.1, 2.3, 2.6 and Corollary 2.4, Lemma 2.5.1 are true in the case $\mathcal{A}_\infty$ with $\mathbf{b} = \mathbf{b}_\infty$ instead of $\mathbf{b} = \mathbf{b}_r$. Definition 2.5 is valuable for $\mathcal{A}_\infty$ also. Theorem 2.7 is true also for $\mathcal{A}_\infty$, since with the help of projections P_r we have $\gamma = \lim_{r\to\infty} P_r(\gamma)$, $P_r(\gamma) \subset U_r$, $\{P_r(\gamma) : r \in \mathbf{N}\}$

converges to γ uniformly on a compact segment $[a,b]\subset \mathbf{R}$, $\gamma:[a,b]\to \mathcal{A}_\infty$, hence the line integral has the continuous unique extension on $C^0_b(U,\mathcal{A}_\infty)$, where $U_r=P_r(U)$. In Remark 2.8 use $l_2(\mathbf{R})^m$ instead of $\mathbf{R^{2^rm}}$ and representing differential forms η over $\mathcal{A}_\infty$ as pointwise limits (or uniform convergence on compact subsets) of differential forms over $\mathcal{A}_r$ while r tends to the infinity, since $z_r\to z$ while r tends to infinity, where $z\in\mathcal{A}_\infty$, $z_r:=P_r(z)$. In general:
$(i)\quad \eta(z,\tilde{z})=\sum_{I,J}\eta_{I,J}\{(d^{\,p_1}z^{\wedge I_1}\alpha_1\wedge...\wedge d^{\,p_n}z^{\wedge I_n}\alpha_n\wedge d^{\,t_1}\tilde{z}^{\wedge J_1}\beta_1\wedge...\wedge d^{\,t_n}\tilde{z}^{\wedge J_n}\beta_n\}_{q(|I|+|J|+2n)}$
is the differential form over $\mathcal{A}_\infty$, where each $\eta_{I,J}(z,\tilde{z})$ is a continuous function on open U_n in $\mathcal{A}^n_\infty$ with values in $\mathcal{A}_\infty$, $I=(I_1,...,I_n)$, $J=(J_1,...,J_n)$, $|I|:=I_1+...+I_n$, $1\le p_1\le p_2\le ...\le p_n\in\mathbf{N}$, $1\le t_1\le t_2\le ...\le t_n\in\mathbf{N}$, $0\le I_k\in\mathbf{Z}$, $0\le J_k\in\mathbf{Z}$, $\alpha_k,\beta_k\in\mathcal{A}_\infty$ are constants for each $k=1,...,n$, $d^{\,p}z^0:=1$, $d^{\,p}\tilde{z}^0:=1$, $n\in\mathbf{N}$, $\pi^l_n(U_l)\subset U_n$ for each $l\ge n$, where $\pi^l_n:\mathcal{A}^l_\infty\to\mathcal{A}^n_\infty$ is the natural projection for each $l\ge n$. The convergence on the right of Formula (i) in the case of infinite series by I or J is supposed relative to the $C^0_b(W,\mathcal{A}^{\wedge *}_\infty)$ topology of uniform convergence on W, where $W=pr-\lim\{U_n,\pi^l_n,\mathbf{N}\}$, $\mathcal{A}^{\wedge *}_\infty$ is supplied with the norm topology inherited from the adjoint space of all poly $\mathbf{R}$-homogeneous $\mathcal{A}_\infty$-additive functionals.

In Remark 2.10 define $\mathcal{A}_{\infty,s,p}$ and use projections $\pi_{s,p,t}$ for each $s\ne p\in\mathbf{b}$. Theorems $2.11, 2.15$ and Corollaries $2.13, 2.15.1$ are transferrable onto $\mathcal{A}_\infty$ by imposing condition of (2^r-1)-connectedness of $P_r(U)=:U_r$ for each $r\ge 3$, considering $\pi_{s,p,t}(U)$ for each $s=i_{2k}$, $p=i_{2k+1}$, $0\le k\in\mathbf{Z}$. Then Definitions $2.12, 2.14$ and Theorem 2.16, Notes $2.17, 3.1$ are valid for $\mathcal{A}_\infty$ also. The validity of Corollary 3.3 follows from the proof of Theorem 3.6.2. We also have instead of 3.4 and 3.5 the following.

3.4'. Corollary. *The function* $\exp(z)$ *on the set* $\mathcal{I}_\infty:=\{z\in\mathcal{A}_\infty: Re(z)=0\}$ *is periodic with the infinite family of generators of periods* $s\in\hat{b}_\infty$ *such that* $\exp(z(1+2\pi n/|z|))=\exp(z)$ *for each* $0\ne z\in\mathcal{I}_\infty$ *and each integer number* n*. If* $z\in\mathcal{A}_\infty$ *is written in the form* $z=2\pi sM$*, where* $M\in\mathcal{I}_\infty$*,* $|M|=1$*, then* $\exp(z)=1$ *if and only if* $s\in\mathbf{Z}$*.*

3.5'. Corollary. *The function* $\exp$ *is the epimorphism from* $\mathcal{I}_\infty$ *on the infinite dimensional unit sphere* $S^\infty(0,1,\mathcal{A}_\infty):=\{z:z\in\mathcal{A}_\infty,|z|=1\}$*.*

3.7. Note. In the noncommutative $\mathcal{A}_r$, $2\le r\le\infty$, case there is the following relation for exp and its (right) derivative:

$$(3.6)\quad \exp(z)'.h=\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}((z^k)h)z^{n-k-1}/n!,$$

where z and $h \in \mathcal{A}_r$. In particular,

$$(3.7) \quad \exp(z)'.v = v \exp(z)$$

for each $v \in \mathbf{R}$, but generally not for all $h \in \mathcal{A}_r$. The function exp is periodic on $\mathcal{A}_r$, hence the inverse function denoted by Ln is defined only locally.

Let at first $2 \le r \in \mathbf{N}$. Consider the space $\mathbf{R^{2^r}}$ of all variables $(w_s : s \in \mathbf{b})$ for which exp is periodic on $\mathcal{A}_r$. The condition $\sum_{s\in\hat{b}} w_s^2 = 1$ defines in $\mathbf{R^{2^r}}$ the unit sphere S^{2^r-2}. The latter has a central symmetry element C for the transformation $C(w_s : s \in \mathbf{b}) = (-w_s : s \in \mathbf{b})$. Consider a subset $P = P_0 \cup \bigcup_{q,j_1,...,j_q} P_{j_1,...,j_q}$ of S^{2^r-2}, where $1 \le q \le 2^{r-1}-1$, of all points characterized by the conditions:

$P_0 := \{(w_s : s \in \hat{b}) \in S^{2^r-2} : w_s \le 0, \forall s \in \hat{b}\}$,

$P_{j_1,...,j_q} := \{(w_s : s \in \hat{b}) \in S^{2^r-2} : w_s \le 0, \forall s \in \hat{b} \setminus \{j_1,...,j_q\}, w_s \ge 0, \forall s \in \{j_1,...,j_q\}\}$, then $P \cup CP = S^{2^r-2}$ and the intersection $P \cap CP$ is (2^r-3)-dimensional over $\mathbf{R}$. This sphere S^{2^r-2} corresponds to the embedding $\theta_1 : (w_s : s \in \hat{b}) \hookrightarrow (0, w_s : s \in \hat{b}) \in \mathbf{R^{2^r}}$. Consider the embedding of $\mathbf{R^{2^r}}$ into $\mathcal{A}_r$ given by $\theta_2 : (w_s : s \in \mathbf{b}) \hookrightarrow \sum_{s\in\mathbf{b}} w_s s \in \mathcal{A}_r$. This yields the embedding $\theta := \theta_2 \circ \theta_1$ of S^{2^r-2} into $\mathcal{A}_r$. Each unit circle with the center 0 in $\mathcal{A}_r$ intersects the equator $\theta(S^{2^r-2})$ of the unit sphere $S^{2^r-1}(0,1,\mathcal{A}_r)$. Join each point $\sum_{s\in\hat{b}} w_s s$ on $\theta(S^{2^r-2})$ with the zero point in $\mathcal{A}_r$ by a line $\{a\sum_{s\in\hat{b}} w_s s : s \in \bar{\mathbf{R}}_+\}$, where $\bar{\mathbf{R}}_+ := \{a \in \mathbf{R} : a \ge 0\}$. This line crosses a circle embedded into $S^{2^r-1}(0,1,\mathcal{A}_r)$, which is a trace of a circle $\{\exp(2\pi a M) : a \in [0,1]\}$ of radius 1 in $\mathcal{A}_r$, where $M = \sum_{s\in\hat{b}} w_s s$. Therefore, $\psi(s) := \exp(v + 2\pi a M)$ as a function of (v,a) for fixed $(w_s : s \in \hat{b}) \in S^{2^r-2}$ defines a bijection of the domain $X \setminus \{aM : a \in \bar{\mathbf{R}}_+\}$ onto its image, where X is $\mathbf{R^2}$ embedded as $(v,a) \hookrightarrow (v + aM) \in \mathcal{A}_r$, where $v \in \mathbf{R}$. This means, that $Ln(z)$ is correctly defined on each subset $X \setminus \{aM : a \in \bar{\mathbf{R}}_+\}$ in $\mathcal{A}_r$. The union $\bigcup_{(w_s:s\in\hat{b})\in P}\{a\sum_{s\in\hat{b}} w_s s : a \in \bar{\mathbf{R}}_+\}$ produces the (2^r-1)-dimensional (over $\mathbf{R}$) subset $Q := Q_0 \cup \bigcup_{q,j_1,...,j_q} Q_{j_1,...,j_q}$, where $Q_0 = \theta(S_0)$, $Q_{j_1,...,j_q} := \theta(S_{j_1,...,j_q})$, $S_0 := \bar{\mathbf{R}}_+ P_0$, $S_{j_1,...,j_q} := \bar{\mathbf{R}}_+ P_{j_1,...,j_q}$, $1 \le q \le 2^{r-1}-1$. Then, on the domain $\mathcal{A}_r \setminus Q$, the function $\exp(z)$ defines a bijection with image $\exp(\mathcal{A}_r \setminus Q)$ and its inverse function $Ln(z)$ is correctly defined on $\mathcal{A}_r \setminus \exp(Q)$. By rotating $\mathcal{A}_r \setminus Q$ one may produce other domains on which Ln can be defined as the univalued function (that is, $Ln(z)$ is one point in $\mathcal{A}_r$), but not on the entire $\mathcal{A}_r$. This means that $Ln(z)$ is a locally bijective function. We have elementary identities $\cos(2\pi - \phi) = \cos(\phi)$ and $\sin(2\pi - \phi) = -\sin(\phi)$ for

each $\phi \in \mathbf{R}$. If $0 < \phi < 2\pi$, then $w_1 \sin(\phi)/\phi = w_2 \sin(2\pi - \phi)/(2\pi - \phi)$ if and only if $w_1 = -\phi w_2/(2\pi - \phi)$. To exclude this ambiguity we put in Formula (3.3) $\phi = |M| \ge 0$ and $w_{i_1} \ge 0$. Therefore, $Ln(\exp(z)) = z$ on $\mathcal{A}_r \setminus Q$, hence using Formulas (3.4, 3.5) we obtain the multivalued function

$$(3.8) \quad Ln(z) = ln(|z|) + Arg(z), \text{ where } Arg(z) := arg(z) + 2\pi a M$$

on $\mathcal{A}_r \setminus \{0\}$, where ln is the usual real logarithm on $(0, \infty)$, $a \in \mathbf{Z}$,

$$|z| \exp(2\pi arg(z)) = z, \quad arg(z) := \sum_{s \in \hat{b}} w_{s,z} s, \quad (w_{s,z} : s \in \hat{b}) \in \mathbf{R}^{\mathbf{2^r-1}},$$

$\sum_{s \in \hat{b}} w_{s,z}^2 \le 1$, $w_{i_1,z} \ge 0$, $M = \sum_{s \in \hat{b}} w_s s$ is any unit vector (that is, $|M| = 1$) in $\mathcal{I}_r$ commuting with $arg(z) \in \mathcal{A}_r$, $arg(z)$ is uniquely defined by such restriction on $(w_s : s \in \hat{b})$, for example, $M = \zeta arg(z)$ for any $\zeta \in \mathbf{R}$, when $arg(z) \ne 0$.

For each fixed $M \in \mathcal{I}_r$ $\exp(aM)$ is a one-parameter family of special transformations of $\mathcal{A}_r$, that is, $\exp(aM)\eta \in \mathcal{A}_r$ for each $\eta \in \mathcal{A}_r$ and $|\exp(aM)| = 1$, where $\mathcal{A}_r$ as the linear space over $\mathbf{R}$ is isomorphic with $\mathbf{R}^{\mathbf{2^r}}$. On the other hand, there are special transformations of $\mathcal{A}_r$ for which $a = \pi/2 + \pi k$, but M is variable with $|M| = 1$, where $k \in \mathbf{Z}$, then $\exp(z) = (-1)^k M$. To each closed curve γ in $\mathcal{A}_r$ there corresponds a closed curve $P_\xi(\gamma)$ in a $\mathbf{R}$-linear subspace $\xi \ni 0$, where P_ξ is a projection on ξ, for example,

$P_{\mathbf{R} \oplus \mathbf{R}s}(z) = (z - s(zs))/2 = w_1 + w_s s$ for $\xi = \mathbf{R} \oplus \mathbf{R}s$,

$P_{\mathbf{R}s \oplus \mathbf{R}p}(z) = sP_{\mathbf{R} \oplus \mathbf{R}s^*p}(s^*z) = [z - p(s^*z(s^*p))]/2 = w_s s + w_p p$

for each $s \ne p \in \hat{b}$. Particular cases of such special transformations also correspond to $w_s = 0$ for some $s \in \hat{b}$ for $M \ne 0$. To each closed curve γ in $\mathcal{A}_r$ and each a and b in $\mathcal{A}_r$ with $ab \ne 0$ there corresponds a closed curve $(a\gamma)b$ in $\mathcal{A}_r$.

Instead of the Riemann two dimensional surface of the complex logarithm function we get the 2^r-dimensional manifold W, that is, a subset of $Y^{\aleph_0} := \prod_{i \in \mathbf{Z}} Y_i$, where $Y_i = Y$ for each i, such that each Y is a copy of $\mathcal{A}_r$ embedded into $\mathcal{A}_r \times \mathbf{R}^{\mathbf{2^r-1}}$ and cut by a $(2^r - 1)$-dimensional submanifold Q and with diffeomorphic bending of a neighbourhood of Q such that two $(2^r - 1)$-dimensional edges ${}_1Q$ and ${}_2Q$ of Y diffeomorphic to Q do not intersect outside zero, ${}_1Q \cap {}_2Q = \{0\}$, that is, the boundary ∂Q is also cut everywhere outside zero. We have $\partial Q = \bigcup_{j \in \hat{b}} \partial Q^j$, where

$\partial Q^j := \{\theta(w_s : s \in \hat{b}) : w_j = 0, (w_s : s \in \hat{b}) \in S_0 \cup \bigcup_{q,j_1,...,j_q} S_{j_1,...,j_q}\}$, for each $j \in \hat{b}$. To exclude rotations in each subspace $v + a\xi$ isomorphic with $\mathbf{R^2}$ and embedded into $\mathbf{R} + \partial Q^j$, where $\xi \in \partial Q^j$, $\xi \ne 0$, we have cut ∂Q. Then in $\mathcal{A}_r \times \mathbf{R^{2^r-1}}$ two copies Y_i and Y_{i+1} are glued by the equivalence relation of ${}_2Q_i$ with ${}_1Q_{i+1}$ via the segments $\{a_{l,i}M : a_{l,i} \in \bar{\mathbf{R}}_+\}$ such that $a_{1,i+1} = a_{2,i}$ for each $a_{l,i} \in \bar{\mathbf{R}}_+$ and each given real $(w_s : s \in \hat{b}) \in P$ with $M = \sum_{s\in\hat{b}} w_s s$, $|M| = 1$. This defines the 2^r-dimensional manifold W embedded into $\mathcal{A}_r \times \mathbf{R^{2^r-1}}$ and $Ln : \mathcal{A}_r \setminus \{0\} \to W$ is the univalued function, that is, $Ln(z)$ is a singleton in W for each $z \in \mathcal{A}_r \setminus \{0\}$.

In the case of $\mathcal{A}_\infty$ consider a family Υ of subsets of $\hat{b} = \hat{b}_\infty$ such that if $A \in \Upsilon$, then $\hat{b} \setminus A \notin \Upsilon$, put $Q := \{M \in \mathcal{I}_\infty : w_s \le 0\ \forall s \in A,\ w_s \ge 0\ \forall s \in \mathbf{b} \setminus A,\ A \in \Upsilon\}$, where $M = \sum_{s\in\hat{b}} w_s s$, $w_s \in \mathbf{R}$ for each $s \in \mathbf{b}$. Then $\partial Q = \bigcup_{j\in\hat{b}} \partial Q^j$, where $\partial Q^j := \{z \in Q : w_j = 0\}$. Consider $Y^{\aleph_0} = \prod_{k\in\mathbf{N}} Y_k$, where $Y_k = Y$ is a copy of $\mathcal{A}_\infty$ embedded into $\mathcal{A}_\infty \times l_2(\mathbf{R})$ and cut by the infinite dimensional submanifold Q and with diffeomorphic bending of a neighbourhood of Q such that infinite dimensional edges ${}_1Q$ and ${}_2Q$ do not intersect outside zero, ${}_1Q \cap {}_2Q = \{0\}$. Then in $\mathcal{A}_\infty \times l_2(\mathbf{R})$ two copies Y_k and Y_{k+1} are glued by the equivalence relation of ${}_2Q_k$ with ${}_1Q_{k+1}$ via the segments $\{a_{l,k}M : a_{l,k} \in \bar{\mathbf{R}}_+\}$ such that $a_{1,k+1} = a_{2,k}$ for each $a_{l,k} \in \bar{\mathbf{R}}_+$ and each given $M = \sum_{s\in\hat{b}} w_s s \in Q \cap S(0,1,\mathcal{I}_\infty)$ with $|M| = 1$, where $S(z_0,\rho,\mathcal{I}_\infty) := \{z \in \mathcal{I}_\infty : |z - z_0| = \rho\}$, $\rho > 0$. This defines the infinite dimensional manifold W embedded into $\mathcal{A}_\infty \times l_2(\mathbf{R})$ and $Ln : \mathcal{A}_\infty \setminus \{0\} \to W$ is the univalued function.

3.8. Remark. In the real case trigonometric and hyperbolic functions are different, but defined as functions of the $\mathcal{A}_r$-variable they are related. Put

$\cos(v) := [\exp(vM) + \exp(-vM)]/2$,
$\sin(v) := [\exp(vM) - \exp(-vM)]M^*/2$,
$\cosh(v) := [\exp(v) + \exp(-v)]/2$,
$\sinh(v) := [\exp(v) - \exp(-v)]/2$

for each $v \in \mathbf{R}$, $M \in \mathcal{I}_r$, $|M| = 1$, $2 \le r \le \infty$ and

$\cos(z) := [\exp(z) + \exp(-z)]/2$ and
$\sin(z) := [\exp(z) - \exp(-z)](z^* - z)/[2|z - z^*|]$ for each $z \in \mathcal{A}_r$, where $z \ne z^*$ in the latter case, then

$\cos(v + yM) = \cos(v)\cosh(y) - \sin(v)\sinh(y)M$,
$\sin(v + yM) = \sin(v)\cosh(y) + \cos(v)\sinh(y)M$

for each $v, y \in \mathbf{R}$ and M as above.

3.8.1. Proposition. *Let suppositions of Proposition* 2.2.1 *be satisfied with* $k = n = m$ *and* $f \circ g(z) = z$ *for each* $z \in U$, *where* $g : U \to W$ *is a bijective surjective mapping, that is, there exists the inverse mapping* $f = g^{-1}$. *Then*

$(Df(g)|_{g=g(z)}).\eta = (Dg(z))^{-1}.\eta$ *for each* $\eta \in \mathcal{A}_r^n$, $2 \le r \le \infty$. *If* f *is either* z *or* $\tilde{z}$ *or* $(z, \tilde{z})$*-superdifferentiable, then* g *is either* z *or* $\tilde{z}$ *or* $(z, \tilde{z})$*-superdifferentiable respectively.*

Proof. In view of Proposition 2.2.1 $(Df(g).((Dg(z)).h) = h$ for each $z \in U$ and each $h \in \mathcal{A}_r^n$. Since $Dg(z)$ is $\mathbf{R}$-homogeneous and $\mathcal{A}_r^n$-additive, then DG has an inverse operator, where $G = g \circ \sigma$ is a diffeomorphism of a real domain P in $\mathbf{R^{2^r n}}$ corresponding to U, $\sigma(P) = U$. Therefore, there exists an inverse $\mathbf{R}$-homogeneous $\mathcal{A}_r^n$-additive operator $(Dg(z))^{-1}$ on $\mathcal{A}_r^n$. Putting $\eta = (Dg(z)).h$ we get the statement of this proposition. The latter statement of this proposition follows from Proposition 2.3, since g is either z or $\tilde{z}$ or $({}_1z, {}_2z)|_{{}_1z=z, {}_2z=\tilde{z}}$-represented together with f.

3.8.2. Proposition. *Let* $g : U \to \mathcal{A}_r$, $\infty \ge r \ge 3$, *be an* $\mathcal{A}_r$*-holomorphic function on* U, *where* U *is open in* $\mathcal{A}_r$. *If* $0 \notin g(U)$, *then there exists the* $\mathcal{A}_r$*-holomorphic function* $f = 1/g$ *on* U *such that*

$(Df(z)).h = [D(1/g)|_{g=g(z)}].((Dg(z)).h)$ *for each* $h \in \mathcal{A}_r$.

In particular, $(Df(z)).h = -\xi[((Dg(z)).h)\xi]$ *for* $r = 3$, $\mathcal{A}_3 = \mathbf{K}$, *where* $\xi := 1/g$.

Proof. In view of Formulas $(3.2, 3.3)$ there exists $f = 1/g$, that is $f(z)g(z) = g(z)f(z) = 1$ for each $z \in U$, but it does not mean an existence of a solution of the equation $(ab)z = a$ in general in $\mathcal{A}_r$ for $r \ge 4$. Using Proposition 2.2.1 we get

$(Df(z)).h = (D(1/g)|_{g=g(z)}).((Dg(z)).h)$,

such that $g((D(1/g)).h) = -((Dg).h)/g = -h/g$ for each $h \in \mathcal{A}_r$. That is $D(1/g)$ exists. If $r = 3$, $\mathcal{A}_3 = \mathbf{K}$, then $(D(1/g)).h = -\xi(h\xi)$, where $\xi := 1/g$, since $\mathbf{K}$ is alternative.

3.8.3. Theorem. *The function* Ln *is* $\mathcal{A}_r$*-holomorphic on any domain* U *in* $\mathcal{A}_r$ *obtained by an* $\mathcal{A}_r$*-holomorphic diffeomorphism of* $\mathcal{A}_r \setminus Q$ *onto* U, *where* $2 \le r \le \infty$. *Each path* γ *in* $\mathcal{A}_r$ *such that* $\gamma(t) = \rho \exp(2\pi t n M)$ *with* $t \in [0, 1]$, $n \in \bar{\mathbf{R}}_+$, $M \in \mathcal{I}_r$, $|M| = 1$ *is closed in* $\mathcal{A}_r$ *if and only if* $n \in \mathbf{N}$, *where* $\rho > 0$. *In this case*

$$(3.9) \qquad \int_\gamma z^{-1}dz = \int_\gamma d(Lnz) = 2\pi n M.$$

Proof. In view of Formulas $(3.2, 3.3)$ each $0 \neq z \in \mathcal{A}_r$ has z^{-1}. If U and V are two open subsets in $\mathcal{A}_r$ and $g : V \to U$ is an $\mathcal{A}_r$-holomorphic diffeomorphism of V onto U and f is an $\mathcal{A}_r$-holomorphic function on V, then $f \circ g^{-1}$ is $\mathcal{A}_r$-holomorphic function on U, since $(f \circ g^{-1})'(z).h = (f'(\zeta)|_{\zeta = g^{-1}(z)}.(g^{-1}(z))'.h$ for each $z \in U$ and each $h \in \mathcal{A}_r$ (see Propositions 2.2.1 and 3.8.1). Since $\exp$ is the diffeomorphism from $\mathcal{A}_r \setminus Q$ onto $\mathcal{A}_r \setminus \exp(Q)$, we have that Ln is $\mathcal{A}_r$-holomorphic on $\mathcal{A}_r \setminus Q$ due to Propositon 3.8.1 and on each of its $\mathcal{A}_r$-holomorphic images after choosing a definite branch of the multivalued function $Ln(z)$ (see Formula (3.8)).

A path γ is defined for each $t \in \mathbf{R}$ not only for $t \in [0, 1]$ due to the existence of exp. In view of Formulas $(3.2, 3.3)$ a path γ is closed (that is, $\gamma(t_0) = \gamma(t_0 + 1)$ for each $t_0 \in \mathbf{R}$) if and only if $\cos(2\pi n) = \cos(0) = 1$ and $\sin(2\pi n) = 0$, that is, $n \in \mathbf{N}$.

From the definition of the line integral we get the equality: $\int_\gamma d(Lnz) = \int_0^1 (Lnz)'.(\gamma'(t)dt)$. Considering integral sums by partitions P of $[0, 1]$ and taking the limit by the family of all P we get, that $\int_\gamma d(Lnz) = Arg(\gamma(1)) - Arg(\gamma(0))$ for a chosen branch of the function $Arg(z)$ (see Formula (3.8)). Therefore, $\int_\gamma d(Lnz) = 2\pi n M$.

Since $\mathcal{A}_r$ is power-associative, then z commutes with itself and we have: $\exp(z)'.z = \exp(z)z$. Therefore, $\exp(Ln(z))'.1 = (d\exp(\eta)/d\eta)|_{\eta = Ln(z)}.(Ln(z))'.1 = \exp(Ln(z))(Ln(z))'.1$, consequently, $(Ln(z))'.1 = \exp(-Ln(z)) = z^{-1}$ and inevitably

$$\lim_P I(z^{-1}, \gamma; P) = \lim_P \sum_l \hat{z}_l^{-1} \Delta z_l = \lim_P \Delta Ln(z_l) = \int_\gamma dLn(z),$$

hence $\int_\gamma z^{-1} dz = \int_\gamma dLn(z)$. That is, $\int_\gamma dLn(z)$ can be considered as the definition of $\int_\gamma z^{-1} dz$.

3.8.4. Notation. Denote an ordered compositions of functions $\{f_1 \circ f_2 \circ ... f_m\}_{q(m)}$, where $q(m) := (q_m, ..., q_3)$, $q_m \in \mathbf{N}$ means that the first (the most internal bracket of) composition is $f_{q_m} \circ f_{q_{m+1}} = f_{q_m}(f_{q_{m+1}}) =: [f_{q_m} \circ f_{q_{m+1}}]$ such that to the situation
$(f_1 \circ ... \circ [f_t \circ f_{t+1}] \circ ... \circ [f_w \circ f_{w+1}] \circ ... \circ f_m)$ with two simultaneous independent compositions, but $t < w$ by our definition of ordering there corresponds $q_m = w$ (apart from the multiplication). After the first composition we get the composition of $({}'f_1 \circ ... \circ {}'f_{m-1})$, where not less than $(m-2)$ elements here

are the same as in the first composition, then q_{m-1} corresponds to the first composition in this new ordered family of functions, and so on by induction from j to $j-1$, $j=m, m-1, ..., 3$. Since q_2 and q_1 are unique, we omit them. After steps $q_m, ..., q_{m-j+1}$ let the corresponding composition be denoted by $\{ {}^jg_1 \circ ... \circ {}^jg_{m-j}\}_{q(m-j)}$, where
${}^jg_1, ..., {}^jg_{m-j}$ are resulting composites on preceding steps, some of them may belong to the set $\{f_1, ..., f_m\}$. If f_l is in the composite on the $k=k(l)$ step, then there may be two variants: $f_l \circ {}^lg_p$ or ${}^lg_p \circ f_l$, where $p=p(l)$, $j=j(l)=k(l)-1$, ${}^0g_p := f_p$. In the first case suppose that $dom(f_l) \subset {}^jg_p(dom\, {}^jg_p)$, in the second case let $dom({}^jg_p) \subset f_l(dom(f_l))$ for each $l=1, ..., m$.

3.8.5. Proposition. *Let f_1,...,f_m be a family of $\mathcal{A}_r$ superdifferentiable functions either all by z or all by $\tilde{z}$ or all by $(z, \tilde{z})$, $f_j : U_j \to \mathcal{A}_r^{t(j)}$, U_j is open in $\mathcal{A}_r^{t(j+1)}$, $t(j) \in \mathbf{N}$ for each $j=1, ..., m$ such that their domains satisfy conditions above, where $2 \le r \le \infty$. Then*

$$(i) \quad (D\{f_1 \circ f_2 \circ ... \circ f_m\}_{q(m)}(z)).h = \{Df_1({}^{j(1)}g_{p(1)}).Df_2({}^{j(2)}g_{p(2)})....(Df_m(z)).h\}_{q(m)}$$

for each $h \in \mathcal{A}_r^{t(m+1)}$, where $Df_l({}^{j(l)}g_{p(l)}).\xi = (Df_l(\eta)|_{\eta={}^{j(l)}g_{p(l)}(z)}).\xi$ for each $\xi \in \mathcal{A}_r^{t(l+1)}$. Moreover, $\{f_1 \circ f_2 \circ ... \circ f_m\}_{q(m)}$ is superdifferentiable either by z or by $\tilde{z}$ or by $(z, \tilde{z})$ correspondingly. If $r=2$, $\mathcal{A}_2 = \mathbf{H}$, then the composition in (i) is associative, for each $r \ge 3$ it can be in general nonassociative. At a marked point $z=a \in U_m$ it takes the form:

$$(ii) \quad (D\{f_1 \circ f_2 \circ ... \circ f_m\}_{q(m)}(z)).h = \{Df_1(\eta_1).Df_2(\eta_2)....(Df_m(z)).h\}_{q(m)},$$

where $\eta_l := f_l(f_{l+1}(...f_{m-1}(f_m(z))...))$ for each $l=1, ..., m-1$.

Proof. For $m=2$ this proposition was proved in §2.2.1. Prove this proposition by induction and applying Proposition 2.2.1 to pairs of functions in appearing compositions. At first mention that the order of composition for the differential is essential for $\mathcal{A}_r$ with $r \ge 3$. The particular case of all right superdifferentiable functions shows, that in general $(Df_1.Df_2).Df_3$ is not equal to $Df_1.(Df_2.Df_3)$, since these operators are right superlinear, but multiplication of matrices with entries in the Cayley-Dickson algebra $\mathcal{A}_r$ with $r \ge 3$ is not associative. Moreover, these expressions may be different, when Df_j are not right superlinear, but only $\mathcal{A}_r^{t(j+1)}$-additive.

Let proposition be proved for all $n \le m$, consider $\{f_1 \circ ... \circ f_{m+1})_{q(m+1)}$. In it f_1 is in the composition on the $k(1)$ step with ${}^{j(1)}g_{p(1)}$ of the type $f_1 \circ {}^{j(1)}g_{p(1)}$, since f_1 is in the extreme left position. If $k(1)=1$, then

${}^{j(1)}g_{p(1)} = f_2$ and
$\{f_1 \circ ... \circ f_{m+1}\}_{q(m+1)} = \{[f_1 \circ f_2] \circ ... \circ f_{m+1}\}_{q(m)}$.
Applying supposition of induction to composition of functions $[f_1 \circ f_2]$, f_3,...,f_{m+1} and then substituting to it expression of $D(f_1 \circ f_2)$ by Proposition 2.2.1 we get the statement of this proposition in the case $k(1) = 1$. If $k(1) > 1$, then on the $k(1)$ step the composition of f_1 and ${}^{j(1)}g_2$,..., ${}^{j(1)}g_{m+1-j(1)}$ is considered, where $j(1) = k(1) - 1 \ge 1$, hence $m + 1 - j(1) \le m$. Applying supposition of induction to $\{f_1 \circ {}^{j(1)}g_2 \circ ... \circ {}^{j(1)}g_{m+1-j(1)}\}_{q(m+1-j(1))}$ and then to each ${}^{j(1)}g_p$ while being a nontrivial composition we get the statement of proposition in the case $k(1) > 1$. The last statement also follows by the considered above induction from Proposition 2.2.1.

Set-theoretical composition of functions is independent from brackets, but it depends only on order of functions: $(f_1 \circ f_2) \circ f_3(z) = (f_1 \circ f_2)(f_3(z)) = f_1(f_2(f_3(z))) = f_1 \circ (f_2 \circ f_3(z))$, etc. by induction. A non-associativity in general appears after superdifferentiation over $\mathcal{A}_r$ with $r \ge 3$. Since $\{f_1 \circ f_2 \circ ... \circ f_m\}_{q(m)}(z) = f_1(f_2(...(f_m(z))...))$, then for a marked point $z = a \in U_m$ Formula (i) takes the form (ii).

In the case of $r = 2$, $\mathcal{A}_2 = \mathbf{H}$, each $\mathbf{R}$-homogeneous $\mathbf{H}$-additive operator A has the form

$A.h = \sum_j B_j h C_j$ for each $h \in \mathbf{H^n}$, where the sum by j is finite, $1 \le j \le 4$ (see formulas in the proof of Theorem 3.28 [27]), where B_j is the $n \times n$ matrix, h is the $n \times 1$ matrix, A_j is the 1×1 matrix with entries in $\mathbf{H}$. Let A_k be an operator corresponding to $Df_k(\eta_k)$ for a given marked point $z = a \in U_3$, $A_k.h = \sum_j B_{j,k} h C_{j,k}$ for each $k = 1, 2, 3$, then $(A_1.A_2).A_3.h = \sum_{j_1,j_2,j_3} (B_{j_1,1} B_{j_2,2})(B_{j_3,3} h C_{j_3,3})(C_{j_2,2} C_{j_1,1}) = \sum_{j_1,j_2,j_3} B_{j_1,1}(B_{j_2,2} B_{j_3,3}) h (C_{j_3,3} C_{j_2,2}) C_{j_1,1}$ $= (A_1.(A_2.A_3)).h$, since the matrix multiplication over $\mathbf{H}$ is associative. Applying the latter formula by induction, we get, that in the case of $\mathbf{H}$ the composition in Formula (i) is associative.

3.9. Theorem. *Let f be an $\mathcal{A}_r$-holomorphic function on an open domain U in $\mathcal{A}_r$, $\infty \ge r \ge 3$. If $(\gamma + z_0)$ and ψ are presented as piecewise unions of paths $\gamma_j + z_0$ and ψ_j with respect to parameter $\theta \in [a_j, b_j]$ and $\theta \in [c_j, d_j]$ respectively with $a_j < b_j$ and $c_j < d_j$ for each $j = 1, ..., n$ and $\bigcup_j [a_j, b_j] = \bigcup_j [c_j, d_j] = [0, 1]$ homotopic relative to $U_j \setminus \{z_0\}$, where $U_j \setminus \{z_0\}$ is a $(2^r - 1)$-connected open domain in $\mathcal{A}_r$ such that $\pi_{s,p,t}(U_j \setminus \{z_0\})$ is simply connected in $\mathbf{C}$ for each $s = i_{2k}$, $p = i_{2k+1}$, $k = 0, 1, ..., 2^{r-1} - 1$ ($\forall 0 \le k \in \mathbf{Z}$ and $P_m(U_j \setminus \{z_0\})$ is $(2^m - 1)$-connected for each $4 \le m \in \mathbf{N}$ if $r = \infty$), each $t \in \mathcal{A}_{r,s,p}$ and $u \in \mathbf{C}_{s,p}$ for which there exists $z = t + u \in \mathcal{A}_r$. If $(\gamma + z_0)$*

and ψ are closed rectifiable paths (loops) in U such that $\gamma(\theta) = \rho \exp(2\pi\theta M)$ with $\theta \in [0,1]$ and a marked $M \in \mathcal{I}_r$, $|M| = 1$ and $z_0 \notin \psi$. Then

$$(3.10) \quad (2\pi) f(z) M = \int_\psi f(\zeta)(\zeta - z)^{-1} d\zeta$$

for each $z \in U$ such that $|z - z_0| < \inf_{\zeta \in \psi([0,1])} |\zeta - z_0|$. If $\mathcal{A}_r$ is alternative, that is, $r = 3$, $\mathcal{A}_3 = \mathbf{K}$, or $f(z) \in \mathbf{R}$, then

$$(3.11) \quad f(z) = (2\pi)^{-1} (\int_\psi f(\zeta)(\zeta - z)^{-1} d\zeta) M^*.$$

Proof. Join γ and ψ by a rectifiable path ω such that $z_0 \notin \omega$, which is going in one direction and the opposite direction, denoted ω^-, such that $\omega_j \cup \psi_j \cup \gamma_j \cup \omega_{j+1}$ is homotopic to a point relative to $U_j \setminus \{z_0\}$ for suitable ω_j and ω_{j+1}, where ω_j joins $\gamma(a_j)$ with $\psi(c_j)$ and ω_{j+1} joins $\psi(d_j)$ with $\gamma(b_j)$ such that z and $z_0 \notin \omega_j$ for each j. Then $\int_{\omega_j} f(\zeta)(\zeta - z)^{-1} d\zeta = -\int_{\omega_j^-} f(\zeta)(\zeta - z)^{-1} d\zeta$ for each j. In view of Theorem 2.15 there is the equality $-\int_{\gamma^- + z} f(\zeta)(\zeta - z)^{-1} d\zeta = \int_\psi f(\zeta)(\zeta - z)^{-1} d\zeta$. Since $\gamma + z$ is a circle around z its radius $\rho > 0$ can be chosen so small, that $f(\zeta) = f(z) + \alpha(\zeta, z)$, where α is a continuous function on U^2 such that $\lim_{\zeta \to z} \alpha(\zeta, z) = 0$, then $\int_{\gamma + z} f(\zeta)(\zeta - z)^{-1} d\zeta = \int_{\gamma + z} f(z)(\zeta - z)^{-1} d\zeta + \delta(\rho) = 2\pi f(z) M + \delta(\rho)$, where $|\delta(r)| \le |\int_\gamma \alpha(\zeta, z)(\zeta - z)^{-1} d\zeta| \le 2\pi \sup_{\zeta \in \gamma} |\alpha(\zeta, z)| C_1 \exp(C_2 \rho^6)$, where C_1 and C_2 are positive constants (see Inequality (2.7.4)), hence there exists $\lim_{\rho \to 0, \rho > 0} \delta(\rho) = 0$. Taking the limit while $\rho > 0$ tends to zero yields the conclusion of this theorem. If $r = 3$, that is, $\mathcal{A}_r = \mathbf{K}$, or $f(z) \in \mathbf{R}$, then $((2\pi) f(z) M) M^* = 2\pi f(z)$.

3.9.1. Corollary. *Let f, U, ψ, z and z_0 be as in Theorem 3.9, then*

$$|f(z)| \le \sup_{(\zeta \in \psi, h \in \mathcal{A}_r, |h| \le 1)} |\hat{f}(\zeta).h|.$$

3.9.2. Theorem. *Let $\{f_n : n \in \mathbf{N}\}$ be a sequence of $\mathcal{A}_r$-holomorphic functions on a neighbourhood W of bounded canonical closed subset U in $\mathcal{A}_r$, $2 \le r \le \infty$, such that $Int(U)$ satisfies conditions of Theorem* 3.9 *and there exists $\delta > 0$ for which $\{f_n : n \in \mathbf{N}\}$ converges uniformly on the δ-neighbourhood of the topological boundary $Fr(U)^\delta$ of U. Then $\{f_n : n \in \mathbf{N}\}$ converges uniformly on U to an $\mathcal{A}_r$-holomorphic function on $Int(U)$.*

Proof. In view of §2.7 the sequence $\hat{f}_n$ is uniformly converging on $Fr(U)^\delta \cap W$. Consider sections of W by planes $i_{2m}\mathbf{R} \oplus i_{2m+1}\mathbf{R}$ for each $m = 0, 1, 2, ...$ and rectifiable loops γ in $Fr(U)^\delta \cap W \cap (i_{2m}\mathbf{R} \oplus i_{2m+1}\mathbf{R})$. For $r > 3$ consider all possible embeddings of $\mathbf{K}$ into $\mathcal{A}_r$ and such copies of $\mathbf{K}$ contain all corresponding loops γ. In view of Estimate 2.7.(4) and Theorem 3.9 the sequence $\{f_n : n \in \mathbf{N}\}$ converges uniformly on U to a holomorphic function, since U is of finite diameter.

3.10. Theorem. *Let f be a continuous function on an open subset U of $\mathcal{A}_r$, $3 \le r \le \infty$. If f is $\mathcal{A}_r$-integral holomorphic on U, then f is $\mathcal{A}_r$ locally z-analytic on U.*

Proof. Let $z_0 \in U$ be a marked point and let Γ denotes the family of all rectifiable paths $\gamma : [0, 1] \to U$ such that $\gamma(0) = z_0$, then $U_0 = \{\gamma(1) : \gamma \in \Gamma\}$ is a connected component of z_0 in U. Therefore, $g = \{\gamma(1), \int_\gamma f(z)dz\}$ is the function with the domain U_0. From Formulas $(3.2, 3.3)$ it follows, that each $0 \ne z \in \mathcal{A}_r$ has an inverse $z^{-1}z = zz^{-1} = 1$ (this does not mean an existence of a solution of the equation $(ab)z = a$ with $b \ne 0$ in general in $\mathcal{A}_r$ for $r \ge 4$). In view of Proposition 2.2.1 $\partial_z(\zeta - z)^{-k}.1 = k(\zeta - z)^{-k-1}$, since $0 = (\partial_z 1).h = (\partial_z z^k z^{-k}).h = ((\partial_z z^k).h)z^{-k} + z^k((\partial_z z^{-k}).h)$ for each $h \in \mathcal{A}_r$ and $z \ne 0$, $z \in \mathcal{A}_r$. As in §2.15 it can be proved, that $F(z) := \int_\gamma f(z)dz$, for each rectifiable γ in U, depends only on initial and final points. This integral is finite, since $\gamma([0, 1])$ is contained in a compact canonical closed subset $W \subset U$ on which f is bounded. Therefore, $(\partial \int_{z_0}^z f(\zeta)d\zeta/\partial z).h = \hat{f}(z).h$ for each $z \in U$ and $h \in \mathcal{A}_r$, $(\partial \int_{z_0}^z f(\zeta)d\zeta/\partial \tilde{z}) = 0$ for each $z \in U$ and $h \in \mathcal{A}_r$, where z_0 is a marked point in U such that z and z_0 are in one connected component of U, since $\int_{z_0}^{z+\Delta z} f(\zeta)d\zeta - \int_{z_0}^z f(\zeta)d\zeta = \hat{f}(z).\Delta z + \epsilon(\Delta z)|\Delta z|$, where $\lim_{\Delta z \to 0} \epsilon(\Delta z) = 0$ (see §2.5). In particular, $\hat{f}(z).1 = f(z)$ for each $z \in U$. Here $\hat{f}$ is correctly defined for each $f \in C^{1,0}(U, \mathcal{A}_r)$ (see Corollary 2.15.1) by continuity of the differentiable integral functional on $C^0(U, \mathcal{A}_r)$. For a given $z \in \mathbf{U}$ choose a neighbourhood W satisfying the conditions of Theorem 3.9. Then there exists a rectifiable path $\psi \subset W$ such that $f(z)$ is presented by Formula (3.10). The latter integral is infinite differentiable by z such that

$$(3.12) \quad 2\pi((\partial^k F(z)/\partial z^k).h)M = (\int_\psi F(\zeta)(\partial^k(\zeta - z)^{-1}/\partial z^k).h)d\zeta),$$

where $h \in \mathcal{A}_r^k$, particularly, for $h = (1, ..., 1) =: 1^{\otimes k}$:

$$(3.13) \quad 2\pi((\partial^k F(z)/\partial z^k).1^{\otimes k})M = k!(\int_\psi F(\zeta)(\zeta - z)^{-k-1} d\zeta),$$

where $M \in \mathcal{I}_r$, $|M| = 1$. For simplicity of notation we can omit $1^{\otimes k}$ on the left of (3.13). In particular, we may choose a ball $W = B(a, R, \mathcal{A}_r) := \{\xi \in \mathcal{A}_r : |\xi - a| \le R\} \subset U$ for a sufficiently small $R > 0$ and $\psi = \gamma + a$, where $\gamma(s) = \rho \exp(2\pi t M)$ with $t \in [0, 1]$, $0 < \rho < R$. If we prove that $F(z)$ is $\mathcal{A}_r$ locally z-analytic, then evidently its z-derivative $f(z)$ will also be $\mathcal{A}_r$ locally z-analytic. If $r \ge 4$, then there exist different embeddings of the octonion algebra $\mathbf{K}$ into $\mathcal{A}_r$ (see §3.6.2). Suppose there is a series $f(z) := \sum_j a_j (z - z_0)^j b_j$ converging for $|z - z_0| < \rho$, where a_j, b_j for each j belong to the same $\Upsilon_a \hookrightarrow \mathbf{C}$, $0 \ne a \in \mathcal{A}_r$, expansion coefficients do not depend on the embedding of $\Upsilon_{a,z-z_0}$ into $\mathcal{A}_r$, while $z - z_0$ is varying within the same copy of $\mathbf{K}$. Then this expansion is valuable for each $z \in \mathcal{A}_r$ with $|z - z_0| < \rho$, since different embeddings of $\mathbf{K}$ (with generators $\{1, M_1, ..., M_7\} \hookrightarrow \mathcal{A}_r$ such that $|M_i| = 1$, $Re(M_i M_j) = 0$ and $|M_i M_j| = |M_i||M_j|$ for each $i \ne j$) into $\mathcal{A}_r$ give all possible values of $z \in \mathcal{A}_r$ with $|z - z_0| < \rho$. Consider ψ such that $\Upsilon_{\zeta-a,z-a}$ has an embedding into $\mathbf{K}$ for each $\zeta \in \psi$, which is not finally restrictive due to Theorem 2.15. In view of the latter statement using the alternative property of $\mathbf{K}$ we consider $z \in B(a, \rho', \mathcal{A}_r)$ with $0 < \rho' < \rho$, then $|z - a| < |\zeta - a|$ for each $\zeta \in \psi$ and $(\zeta - a - (z - a))^{-1} = (1 - (\zeta - a)^{-1}(z - a))^{-1}(\zeta - a)^{-1} = \sum_{k=0}^\infty ((\zeta - a)^{-1}(z - a))^k (\zeta - a)^{-1}$, where $0 \notin \psi$. Therefore,

$$(3.14) \quad 2\pi F(z) M = \sum_{k=0}^\infty \phi_k(z),$$

$$\text{where } \phi_k(z) := (\int_\psi F(\zeta)((\zeta - a)^{-1}(z - a))^k (\zeta - a)^{-1} d\zeta).$$

Thus $|\phi_k(z)| \le \sup_{\zeta \in \psi} |F(\zeta)|(\rho'/\rho)^{-k}$ for each $z \in B(a, \rho', \mathcal{A}_r)$ and series (3.14) converges uniformly on $B(a, \rho', \mathcal{A}_r)$. Each function $\phi_k(z)$ is evidently $\mathcal{A}_r$ locally z-analytic on $B(a, \rho', \mathcal{A}_r)$, hence $F(z)M$ is such too. Since for each $a \in U$ there is an $\rho' > 0$, for which the foregoing holds, it follows that $F(z)M$ is the $\mathcal{A}_r$ locally z-analytic function. Now write $f(z) = \sum_{s \in \mathbf{b}} f_s s$, where $f_s \in \mathbf{R}$ for each $s \in \mathbf{b}$. If the loop γ is nondegenerate, then loops $s\gamma$, γs, $(2^r - 2)^{-1}\{-\gamma + \sum_{s \in \hat{b}} s(\gamma s^*)\} = \tilde{\gamma}$ are nondegenerate. If (i) $\int_\gamma f(\zeta) d\zeta = 0$, then $\int_\gamma s f(\zeta) d\zeta = 0$ and $\int_\gamma f(\zeta) s d\zeta = 0$ (see Theorem 2.7). In view of

Formulas 2.8.(2):
$f_1 = (f + (2^r - 2)^{-1}\{-f + \sum_{s\in \hat{b}_r} s(fs^*)\})/2$ and
$f_p = (i_p(2^r - 2)^{-1}\{-f + \sum_{s\in \hat{b}_r} s(fs^*)\} - fi_p)/2$
for each $i_p \in \hat{b}_r$ (for $r = \infty$ use $\lim_{r\to\infty}$ on the right of the latter formulas) we have that Condition (i) is equivalent to: $\int_\gamma f_s(\zeta)d\zeta = 0$ for each $s \in \mathbf{b}$. Therefore, the proof above shows, that each function $2\pi f_s(z)M$ is $\mathcal{A}_r$ locally z-analytic, where M is arbitrary in $\mathcal{I}_r$, $|M| = 1$. But $f_s(z) \in \mathbf{R}$ for each $z \in U$, hence $f_s(z) = (2\pi f_s(z)M)(2\pi)^{-1}M^* = f_s(z)$ is $\mathcal{A}_r$ locally z-analytic for each $s \in \mathbf{b}$, hence f is also $\mathcal{A}_r$ locally z-analytic on U.

3.11. Note. Theorems 2.11, 2.15, 2.16, 3.10 and Corollary 2.13 establish the equivalence of notions of $\mathcal{A}_r$-holomorphic, $\mathcal{A}_r$-integral holomorphic and $\mathcal{A}_r$ locally z-analytic classes of functions on domains satisfying definite conditions.

3.11.1. Definitions. Let U be an open subset in $\mathcal{A}_r$, $2 \le r \le \infty$ and $f \in C^0(U, \mathcal{A}_r)$, then we say that f possesses a primitive $g \in C^1(U, \mathcal{A}_r)$ if $g'(z).1 = f(z)$ for each $z \in U$. A region U in $\mathcal{A}_r$ is said to be $\mathcal{A}_r$-holomorphically simply connected if every function $\mathcal{A}_r$-holomorphic on it possesses a primitive.

From §3.10 we get.

3.11.2. Theorem. *If $f \in C^\omega_z(U, \mathcal{A}_r)$, $2 \le r \le \infty$, where U is $(2^r - 1)$-connected ($P_m(U)$ is $(2^m - 1)$-connected for each $4 \le m \in \mathbf{N}$ for $r = \infty$); $\pi_{s,p,t}(U)$ is simply connected in $\mathbf{C}$ for each $s = i_{2k}$, $p = i_{2k+1}$ in $\mathbf{b}$, $t \in \mathcal{A}_{r,s,p}$ and $u \in \mathbf{C}_{s,p}$ for which there exists $z = t + u \in U$, U is an open subset in $\mathcal{A}_r$, then there exists $g \in C^\omega_z(U, \mathcal{A}_r)$ such that $g'(z).1 = f(z)$ for each $z \in U$.*

3.11.3. Theorem. *Let U and V be $\mathcal{A}_r$-holomoprhically simply connected regions in $\mathcal{A}_r$, $2 \le r \le \infty$ with $U \cap V \ne \emptyset$ connected. Then $U \cup V$ is $\mathcal{A}_r$-holomorphically simply connected.*

3.12. Corollary. *Let U be an open subset in $\mathcal{A}_r^n$, $2 \le r \le \infty$, then the family of all $\mathcal{A}_r$-holomorphic functions $f : U \to \mathcal{A}_r$ has a structure of an $\mathcal{A}_r$-algebra.*

Proof. If $f_1(z) = \alpha g(z)\beta + \gamma h(z)\delta$ or $f_2(z) = g(z)h(z)$ for each $z \in U$, where α, β, γ and $\delta \in \mathcal{A}_r$ are constants, g and h are $\mathcal{A}_r$-holomorphic functions on U, then F_1 and F_2 are Frechét differentiable on U by $(w_s : s \in \mathbf{b})$ (see §2.1 and §2.2) and $\partial_{\tilde{z}} f_1(z) = \alpha(\partial_{\tilde{z}} g)\beta + \gamma(\partial_{\tilde{z}} h)\delta = 0$ and $\partial_{\tilde{z}} f_2(z) = (\partial_{\tilde{z}} g)h + g(\partial_{\tilde{z}} h) = 0$, hence f_1 and f_2 are also $\mathcal{A}_r$-holomorphic on U.

3.13. Proposition. *For each complex holomorphic function f in a*

neighbourhood $Int(B(q_0,\rho,\mathbf{C}))$, $\infty>\rho>0$, *of a point* $q_0\in\mathbf{C}$ *and each* $2\le r\le\infty$, $s\ne p\in\mathbf{b}_r$, *there exists an* $\mathcal{A}_r$ *z-analytic function* g *on a neighbourhood* $Int(B(a,\rho,\mathcal{A}_r))$ *of* $a\in\mathcal{A}_r$ *such that* $s^*g_{s,p}(u,t_0)=f(v)$ *on* $Int(B(u_0,\rho,\mathbf{C}))$, $u=s(Re(v)+s^*(pIm_{\mathbf{C}}(v)))$, *where* $B(x,\rho,X):=\{y\in X: d_X(x,y)\le\rho\}$ *is the ball in a space* X *with a metric* d_X, $a=u_0+t_0$, $u_0\in\mathbf{C}_{s,p}$, $t_0\in\mathcal{A}_{r,s,p}$, $u_0=s(Re(q_0)+s^*(pIm_{\mathbf{C}}(q_0)))$, $Im_{\mathbf{C}}(v):=(v-\tilde{v})/(2i)$.

Proof. Among Conditions (2.3.1) there are independent:
(3.15) $\quad \partial F_1/\partial\, {}^jw_p=\partial F_{pq^*}/\partial\, {}^jw_q$,
$\partial F_1/\partial\, {}^jw_q=-\partial F_{pq^*}/\partial\, {}^jw_p$
for each $p=i_m$, $q=i_{m+1}\in\mathbf{b}_r$, $0\le m\in\mathbf{Z}$. Consider a homogeneous polynomial function on an open ball $Int(B)$ in $\mathcal{A}_r$ such that $P_n(\lambda z)=\lambda^nP_n(z)$ for each $\lambda\in\mathbf{R}$, $P_n: Int(B)\to\mathcal{A}_r$, then P_{n+1} can be written in the form
(3.16) $\quad P_{n+1}(z)=\sum_{s\in\mathbf{b}_r;k;j_1,...,j_k}C_{s;k;j_1,...,j_k}v_1^{j_1}...v_k^{j_k}s$;
where $v_l:=w_{i_l}$ is the real variable for each l, $z=\sum_{s\in\mathbf{b}_r}w_ss$, $0\le j_1\in\mathbf{Z},...,0\le j_k\in\mathbf{Z}$, $k\in\mathbf{N}$, $j_1+...+j_k=n+1$, $C_{s;k;j_1,...,j_k}\in\mathbf{R}$ is the real expansion coefficient for each $s,k,j_1,...,j_k$. In view of (2.3.1) a function f is right superlinearly $\mathcal{A}_r$-superdifferentiable at a point z_0 if and only if sf is right superlinearly $\mathcal{A}_r$-superdifferentiable at z_0 for each $s\in\hat{b}_r$. Then (3.15) applied to (3.16) gives conditions on coefficients of homogeneous polynomial providing its right superlinear superdifferentiability:
(3.17) $\quad C_{1;k;j_1,...,j_m+1,...,j_k}(j_m+1)=C_{pq^*;k;j_1,...,j_{m+1}+1,...,j_k}(j_{m+1}+1)$ and
$C_{1;k;j_1,...,j_{m+1}+1,...,j_k}(j_{m+1}+1)=-C_{pq^*;k;j_1,...,j_m+1,...,j_k}(j_m+1)$,
for each $p=i_m$, $q=i_{m+1}$ in $\mathbf{b}_r$. Since $j_l+1\ge1$ for each l, then coefficients on the right sides are expressible through coefficients on the left, which can be taken as free variables. Therefore, for each $n\ge0$ there exists nontrivial (nonzero) P_{n+1} satisfying (3.15). Evidently, the $\mathbf{R}$-linear space of all right superlinearly $\mathcal{A}_r$-superdifferentiable functions is infinite dimensional, since for each n there exists a nontrivial solution of the homogeneous system of linear equations (3.17).

Consider first an extension in the class of $\mathcal{A}_r$-holomorphic functions with a superdifferential not necessarily right superlinear on the superalgebra $\mathcal{A}_r$. Since f is holomorphic in $Int(B(q_0,\rho,\mathbf{C}))$, it has a decomposition $f(t)=\sum_{n=0}^\infty f_n(q-q_0)^n$, where $f_n\in\mathbf{C}$ are expansion coefficients, $q\in Int(B(q_0,\rho,\mathbf{C}))$. Consider its extension in $Int(B(z_0,\rho,\mathcal{A}_r))$ such that $f(z)=\sum_{n=0}^\infty f_n(z-z_0)^n$, $z_0=u_0+t_0$, $u_0=s(Re(q_0)+s^*(pIm_{\mathbf{C}}(q_0)))$, $t_0\in\mathcal{A}_{r,s,p}$. Evidently this series converges for each $z\in Int(B(z_0,\rho,\mathcal{A}))$ and this extension of f is $\mathcal{A}_r$-holomorphic, since $f_n\in\mathcal{A}_r$ for each n and $\partial f/\partial\tilde{z}=0$.

Consider now more narrow class of quaternion holomorphic functions with a right superlinear superdifferential on the superalgebra $\mathcal{A}_r$. The Cauchy-Riemann conditions for complex holomorphic functions are particular cases (part) of Conditions (3.15). Having grouped the series for complex holomorphic function f in the series by homogeneous polynomials and applying (3.17) we get expansion coefficients for the right superlinear superdifferentiable extension of f on $Int(B(z_0, \rho, \mathcal{A}))$.

3.14. Proposition. *If f is an $\mathcal{A}_r$-holomorphic function on an open subset U in $\mathcal{A}_r$, $2 \le r \le \infty$, and $ker(f'(z_0)) = \{0\}$ and $f'(z_0)$ is right superlinear, then it is a conformal mapping at a marked point $z_0 \in U$, that is preserving angles between differentiable curves. If $r = 3$, $\mathcal{A}_3 = \mathbf{K}$, then $ker(f'(z_0)) = \{0\}$ if and only if $f'(z_0) \ne 0$.*

Proof. Let $z_0 \in U$ and f be right superlinearly superdifferentiable at z_0. Consider a scalar product $(*,*)_r$ in $\mathbf{C}^m$ as in the introduction, $m = 2^{r-1}$ for $r < \infty$ or in $l_2(\mathbf{C})$ inherited from $\mathcal{A}_\infty$, where $(x,y)_{l_2} = \lim_{r\to\infty}(x_r, y_r)_r$ (see §3.6.2 also). To the right superlinear operator $f'(z_0)$ there corresponds the unique bounded operator A on $\mathbf{C}^m$ or $l_2(\mathbf{C})$ respectively such that $ker(A) = \{0\}$, that is, $Ah = 0$ if and only if $h = 0$. If $r = 3$, then A is invertible if and only if $f'(z_0) \ne 0$, since $f'(z_0) \in \mathcal{A}_r$ and $\mathbf{K}$ is alternative: $(ab)y = a$ has the unique solution y for each $b \ne 0$ in $\mathbf{K}$. In view of the polar decomposition (3.5) $f'(z_0) = \rho \exp(M)$, where $\rho > 0$ and $M \in \mathcal{I}_r$. Then the adjoint operator A^* corresponds to $\rho \exp(-M)$, but $\exp(-M)\exp(M) = 1$, hence A is the unitary operator: $A \in U(m)$ or $A \in U(\infty)$ respectively. Since the unitary group preserves scalar product, then $f(z)$ preserves an angle α between each two differentiable curves in U crossing at the marked point z_0: if ψ and $\phi : (-1,1) \to U$ are two differentiable curves crossing at a point $z_0 \in U$, then $f(\psi(\theta))' = f'(z)|_{z=\psi(\theta)}.\psi'(\theta)$, where $\cos(\alpha) = Re(\psi'(0), \phi'(0))/(|\psi'(0)||\phi'(0)|)$ for $\psi'(0) \ne 0$ and $\phi'(0) \ne 0$.

3.14.1. Remark. For each $r \ge 4$ the Cayley-Dickson algebra is not the division algebra, hence the condition $ker(f'(z_0)) = \{0\}$ is essential in Proposition 3.14.

3.15. Theorem. *Let f be an $\mathcal{A}_r$-holomorphic function, $2 \le r \le \infty$, on an open subset U in $\mathcal{A}_r$ such that $\sup_{z\in U, h\in B(0,1,\mathcal{A}_r)} |[f(z)(\zeta - z)^{-2}]\hat{}.h| \le C/|\zeta - z|^2$ for each $\zeta \in \mathcal{A}_r \setminus cl(U)$. Then $|f'(z)| \le C/d(z)$ for each $z \in U$, where $d(z) := \inf_{\zeta \in \mathcal{A}_r \setminus U} |\zeta - z|$; $|f(\xi) - f(z)|/|\xi - z| \le 2C/\rho$ for each ξ and $z \in B(a, \rho/2, \mathcal{A}_r) \subset Int(B(a,\rho,\mathcal{A}_r)) \subset U$, where $\rho > 0$. In particular, if f is an $\mathcal{A}_r$-holomorphic function with bounded $[f(z)(\zeta - z)^{-2}]\hat{}.h|\zeta - z|^2$ on*

$\mathcal{A}_r^2 \times B(0,1,\mathcal{A}_r)$ *with* $|\zeta| \ge 2|z|$*, that is,* $\sup_{\zeta,z\in\mathcal{A}_r,|\zeta|\ge 2|z|,h\in B(0,1,\mathcal{A}_r)} |[f(z)(\zeta - z)^{-2}]\hat{}.h||\zeta - z|^2 < \infty$*, then* f *is constant.*

Proof. In view of Theorem 3.9 there exists a rectifiable path γ in U such that

$$(3.18)\quad (\partial^k f(z)M/\partial z^k).h = (2\pi)^{-1}(\int_{\gamma+z_0} f(\zeta)(\partial^k(\zeta - z)^{-1}/\partial z^k).hd\zeta)$$

for each $h \in \mathcal{A}_r^k$, where $\gamma(t) = \rho' \exp(2\pi tM)$ with $t \in [0,1]$, $0 < \rho'$. Then, in particular, for $h = 1^{\otimes k}$ and omitting it for short, we get

$$(3.19)\quad (\partial^k f(z)M/\partial z^k) = k!(2\pi)^{-1}(\int_{\gamma+z_0} f(\zeta)(\zeta - z)^{-k-1}d\zeta).$$

Therefore, $|f'(z)| \le C/d(z)$, since $|(\partial(\zeta - z)^{-1}/\partial z).s| = |\zeta - z|^{-2}$ for each $s \in \mathbf{b}$. Since $\int_\zeta^z df(z) = f(z) - f(\zeta)$, then $|f(\xi) - f(z)|/|\xi - z| \le \sup_{z\in B(a,\rho/2,\mathcal{A}_r)}[C/d(z)] \le 2C/\rho$, where $\rho' < \rho/2$, ξ and $z \in B(a,\rho/2,\mathcal{A}_r) \subset Int(B(a,\rho,\mathcal{A}_r)) \subset U$. Taking ρ tending to infinity, if f is $\mathcal{A}_r$-holomorphic with bounded $[f(z)(\zeta - z)^{-2}]\hat{}.h|\zeta - z|^2$ on $\mathcal{A}_r^2 \times B(0,1,\mathcal{A}_r)$ for $|\zeta| \ge 2|z|$, then $f'(z) = 0$ for each $z \in \mathcal{A}_r$, since f is locally z-analytic and $\sup_{\zeta,z\in U,|\zeta|\ge 2|z|,h\in B(0,1,\mathcal{A}_r)} |[f(z)(\zeta - z)^{-2}]\hat{}.h||\zeta - z|^2 < \infty$ is bounded, hence f is constant on $\mathcal{A}_r$.

3.16. Remark. Theorems 3.9, 3.10 and 3.15 are the $\mathcal{A}_r$-analogs of the Cauchy, Morera and Liouville theorems correspondingly. Evidently, Theorem 3.15 is also true for right superlinear $\hat{f}(z)$ on $\mathcal{A}_r$ for each $z \in U$ and with bounded $\hat{f}(z).h$ on $U \times B(0,1,\mathcal{A}_r)$ instead of $[f(z)(\zeta - z)^{-2}]\hat{}.h|\zeta - z|^2$.

3.17. Theorem. *Let* $P(z)$ *be a polynomial on* $\mathcal{A}_r$, $2 \le r \le \infty$*, such that* $P(z) = z^{n+1} + \sum_{\eta(k)=0}^n (A_k, z^k)$*, where* $A_k = (a_{1,k}, ..., a_{m,k})$, $a_{j,l} \in \mathcal{A}_r$, $k = (k_1, ..., k_m)$, $0 \le k_j \in \mathbf{Z}$, $\eta(k) = k_1 + ... + k_m$, $0 \le m = m(k) \in \mathbf{Z}$, $m(k) \le \eta(k) + 1$, $(A_k, z^k) := \{a_{1,k}z^{k_1}...a_{m,k}z^{k_s}\}_{q(m+\eta(k))}$, $z^0 := 1$*. Then* $P(z)$ *has a root in* $\mathcal{A}_r$*.*

Proof. Consider at first $r < \infty$. Suppose that $P(z) \ne 0$ for each $z \in \mathcal{A}_r$. Consider a rectifiable path γ_R in $\mathcal{A}_r$ such that $\gamma_R([0,1]) \cap \mathcal{A}_r = [-R,R]$ and outside $[-R,R]$: $\gamma_R(t) = R\exp(2\pi tM)$, where M is a vector in $\mathcal{I}_r$ with $|M| = 1$, $0 \le t \le 1/2$. Express $\tilde{P}$ through variable z also using $z^* = (2^r - 2)^{-1}\{-z + \sum_{s\in\hat{b}_r} s(zs^*)\}$. Since $\lim_{|z|\to\infty} P(z)z^{-n-1} = 1$, then due to Theorem 2.11 $\lim_{R\to\infty}\int_{\gamma_R}(P\tilde{P})^{-1}(z)dz = \int_{-R}^R (P\tilde{P})^{-1}(v)dv = \int_{-R}^R |P(v)|^{-2}dv \ge 0$. But $\lim_{R\to\infty}\int_{\gamma_R}(P\tilde{P})^{-1}(z)dz = \lim_{R\to\infty}\pi R^{-2n-1} = 0$. On the other hand, $\int_{-R}^R |P(v)|^{-2}dv = 0$ if and only if $|P(v)|^{-2} = 0$ for each

$v \in \mathbf{R}$. This contradicts our supposition, hence there exists a root $z_0 \in \mathcal{A}_r$, that is, $P(z_0) = 0$. In the case $r = \infty$ use that $z = \lim_{r\to\infty} z_r$.

3.18. Theorem. *Let f be an $\mathcal{A}_r$-holomorphic function on an open subset U in $\mathcal{A}_r$, $2 \le r \le \infty$. Suppose that $\epsilon > 0$ and K is a compact subset of U. Then for each $M \in \mathcal{I}_r$, $|M| = 1$, there exists a function $g_M(z) = P_\infty(z) + \sum_{k=1}^{\nu} P_k[(z-a_k)^{-1}]$, $z \in \mathcal{A}_r \setminus \{a_1, ..., a_\nu\}$, $\nu \in \mathbf{N}$, where P_∞ and P_j are polynomials, $a_j \in Fr(U)$, $Fr(U)$ denotes a topological boundary of U in $\mathcal{A}_r$, such that $|f(z)M - g_M(z)| < \epsilon$ for each $z \in \mathsf{K}$. If $r = 2$ or $r = 3$, then this statement is true for $M = 1$ also.*

Proof. Consider cubes S_j with ribs parallel to the basic axes and of length n^{-1} in $\mathcal{A}_r$ and putting $S := \cup_j S_j$ such that $\mathsf{K} \subset Int(S)$, where $n \in \mathbf{N}$ tends to infinity. Since f is $\mathcal{A}_r$-holomorphic and $\mathbf{K}$ is compact, we may apply Formula (3.10) to each $\gamma \subset Fr(S_j)$ defined by directing vector $M \in \mathcal{I}_r$, $|M| = 1$. It can be seen, that f can be approximated uniformly on K by a sum of the form $\sum_{k=1}^{\mu} \{(a_{1,k}(\zeta_k - z)^{-1} a_{2,k})\}_{q(3)}$, where $a_{j,k} \in \mathcal{A}_r$, $\zeta_k \in Fr(S_j)$. For a given $n \in \mathbf{N}$ if $b \in Fr(S_j)$, then there exists $a \in Fr(U_j) \cup \partial B(0, \rho, \mathcal{A}_r)$ such that $|b - a| \le n^{-1}$. If $z \in \mathsf{K}$ and $|z - a| \ge n^{-1}$, then the series $(z-b)^{-1} = (\sum_{k=0}^{\infty}[(z-a)^{-1}(b-a)]^k)(z-a)^{-1}$ converges uniformly on K and it is clear that fM can be approximated uniformly on K by a function of the indicated form. In particular, for $r = 2$ or $r = 3$ the equation $(ab)y = a$ has a solution for each $b \ne 0$ in $\mathcal{A}_r$, that gives approximation of f by $g_M M^*$.

3.19. Note and Definitions. Consider a one-point (Alexandroff) compactification $\hat{\mathcal{A}}_r$ of the locally compact topological space $\mathcal{A}_r$ for $2 \le r \in \mathbf{N}$. It is homeomorphic to a unit 2^r-dimensional sphere S^{2^r} in the Euclidean space $\mathbf{R^{2^r+1}}$. In case $r = \infty$ consider a unit sphere S^∞ with centre in 0 in $\mathbf{R} \oplus \mathcal{A}_\infty$ such that $\mathcal{A}_\infty$ is topologically homeomorphic to $S^\infty \setminus \{(1,0,0,...)\}$. If ζ is a point in S^{2^r} or S^∞ different from $(1,0,0,...)$, then the straight line containing $(1,0,0,...)$ and ζ crosses π_S in a finite point z, where π_S is the 2^r-dimensional or ∞-dimensional respectively plane orthogonal to the vector $(1,0,0,...)$ and tangent to S^{2^r} or S^∞ at the south pole $(-1,0,0,...)$. This defines the bijective continuous mapping from $S^{2^r} \setminus \{(1,0,0,...)\}$ or $S^\infty \setminus \{(1,0,0,...)\}$ onto π_S such that $(1,0,0,...)$ corresponds to the point of infinity. Therefore each function on a subset U of $\mathcal{A}_r$ as a topological space can be considered on the homeomorphic subset V in S^{2^r} or S^∞. The infinite dimensional sphere S^∞ is not (locally) compact relative to the norm topology, but it is compact relative to the weak topology inherited from l_2 (see [9, 28]).

Let $z_0 \in \hat{\mathcal{A}}_r$ be a marked point. If a function f is defined and $\mathcal{A}_r$-

holomorphic on $V \setminus \{z_0\}$, where V is a neighbourhood of z_0, then z_0 is called a point of an isolated singularity of f.

Suppose that f is an $\mathcal{A}_r$-holomorphic function in $B(a, 0, \rho, \mathcal{A}_r) \setminus \{a\}$ for some $\rho > 0$. Then we say that f has an isolated singularity at a. Let $B(\infty, \rho, \mathcal{A}_r) := \{z \in \hat{\mathcal{A}}_r$ such that $\rho^{-1} < |z| \le \infty\}$. Then we say that f has an isolated singularity at ∞ if it is $\mathcal{A}_r$-holomorphic in some $B(\infty, \rho, \mathcal{A}_r)$.

Let $f : U \to \mathcal{A}_r$ be a function, where U is a neighbourhood of $z \in \hat{\mathcal{A}}_r$. Then f is said to be meromorphic at z if f has an isolated singularity at z. If U is an open subset in $\hat{\mathcal{A}}_r$, then f is called meromorphic in U if f is meromorphic at each point $z \in U$. If U is a domain of f and f is meromorphic in U, then f is called meromorphic on U. Denote by $\mathbf{M}(U)$ the set of all meromorphic functions on U. Let f be meromorphic on a region U in $\hat{\mathcal{A}}_r$. A point

$$c \in \bigcap_{V \subset U, V \text{ is closed and bounded}} cl(f(U \setminus V))$$

is called a cluster value of f.

3.20. Proposition. *Let f be an $\mathcal{A}_r$-holomorphic function, $2 \le r \le \infty$, with a right $\mathcal{A}_r$-superlinear superdifferential on an open connected subset $U \subset \hat{\mathcal{A}}_r$ and suppose that there exists a sequence of points $z_n \in U$ having a cluster point $z \in U$ such that $f(z_n) = 0$ for each $n \in \mathbf{N}$, then $f = 0$ everywhere on U.*

Proof follows from the local z-analyticity of f and the fact $f^{(k)}(z) = 0$ for each $0 \le k \in \mathbf{Z}$ (see Definiton 2.2, Theorems 2.11 and 3.10), when $f'(z)$ is right $\mathcal{A}_r$-superlinear on U. Therefore, f is equal to zero on a neighbourhood of z. The maximal subset of U on which f is equal to zero is open in U. On the other hand it is closed, since f is continuous, hence f is equal to zero on U, since U is connected.

3.21. Theorem. *Let $\mathbf{A}$ denote the family of all functions f such that f is $\mathcal{A}_r$-holomorphic on $U := Int(B(a, \rho, R, \mathcal{A}_r))$, $3 \le r \le \infty$, where a is a marked point in $\mathcal{A}_r$, $0 \le \rho < R < \infty$ are fixed. Let $\mathbf{S}$ denote a subset of $\mathbf{Z}^{\mathbf{N}}$ such that for each $k \in \mathbf{S}$ there exists $m(k) := \max\{j : k_j \ne 0, k_i = 0 \text{ for each } i > j\} \in \mathbf{N}$ and let $\mathbf{B}$ be a family of finite sequences $b_k = (b_{k,1}, ..., b_{k,m(k)}; q(m(k) + \eta(k)))$ such that $b_{k,j} \in \mathcal{A}_r$ for each $j = 1, ..., n$, $n \in \mathbf{N}$, $q(m(k)+\eta(k))$ as in §2.1. Then there exists a bijective correspondence*

between $\mathbf{A}$ *and* $\zeta \in \mathbf{B}^{\mathbf{S}}$ *such that*

$$(3.20)\quad \lim_{m+\eta\to\infty} \sup_{z\in B(a,\rho_1,R_1,\mathcal{A}_r)} \sum_{k,m(k)=m,\eta(k)=\eta} |\{(b_k, z^k)\}_{q(m(k)+\eta(k))}| = 0$$

for each ρ_1 *and* R_1 *such that* $\rho < \rho_1 < R_1 < R$, *where* $\eta(k) := k_1 + ... + k_{m(k)}$, $\zeta(k) =: b_k = (b_{k,1}, ..., b_{k,m(k)}; q(m(k) + \eta(k)))$, $\{(b_k, z^k)\}_{q(m(k)+\eta(k))} = \{b_{k,1}z^{k_1}...b_{k,m(k)}z^{k_{m(k)}}\}_{q(m(k)+\eta(k))}$ *for each* $k \in \mathbf{S}$, *that is,* $f \in \mathbf{A}$ *can be presented by a convergent series*

$$(3.21)\quad f(z) = \sum_{b\in\zeta} \{(b_k, z^k)\}_{q(m(k)+\eta(k))}.$$

Proof. If Condition (3.20) is satisfied, then the series (3.21) converges on $B(a, \rho', R', \mathcal{A}_r)$ for each ρ' and R' such that $\rho < \rho' < R' < R$, since ρ_1 and R_1 are arbitrary such that $\rho < \rho_1 < R_1 < R$ and $\sum_{n=0}^{\infty} p^n$ converges for each $|p| < 1$. In particular taking $\rho_1 < \rho' < R' < R_1$ for $p = R'/R_1$ or $p = \rho_1/\rho'$. Therefore, from (3.20) and (3.21) it follows, that f presented by the series (3.21) is $\mathcal{A}_r$-holomorphic on U.

Vice versa let f be in $\mathbf{A}$. In view of Theorems 2.11 and 3.9 there are two rectifiable closed paths γ_1 and γ_2 such that $\gamma_2(t) = a + \rho' \exp(2\pi t M_2)$ and $\gamma_1(t) = a + R' \exp(2\pi t M_1)$, where $t \in [0, 1]$, M_1 and $M_2 \in \mathcal{A}_r$ with $|M_1| = 1$ and $|M_2| = 1$, where $\rho < \rho' < R' < R$, because as in §3.9 U can be presented as a finite union of regions U_j each of which satisfies the conditions of Theorem 2.11. Using a finite number of rectifiable paths w_j (joining γ_1 and γ_2 within U_j) going twice in one and the opposite directions leads to the conclusion that for each $z \in Int(B(a, \rho', R', \mathcal{A}_r))$ the function $f(z)M$ with $M = M_1 = M_2$ is presented by the integral formula:

$$(3.22)\quad f(z)M = (2\pi)^{-1}\{(\int_{\gamma_1} f(\zeta)(\zeta - z)^{-1}d\zeta) - (\int_{\gamma_2} f(\zeta)(\zeta - z)^{-1}d\zeta)\}.$$

On γ_1 we have the inequality: $|(\zeta-a)^{-1}(z-a)| < 1$, on γ_2 another inequality holds: $|(\zeta-a)(z-a)^{-1}| < 1$. In view of §3.6.2 as in §3.10 considering different possible embeddings $\Upsilon_{\zeta-a,z-a} \subset \mathbf{K}$ into $\mathcal{A}_r$ and using alternative property of the octonion algebra $\mathbf{K}$ we get, that for γ_1 the series

$$(\zeta - z)^{-1} = (\sum_{k=0}^{\infty}((\zeta - a)^{-1}(z - a))^k)(\zeta - a)^{-1}$$

converges uniformly by $\zeta \in B(a, R_2+\epsilon, R_1, \mathcal{A}_r)$ and $z \in B(a, \rho_2, R_2, \mathcal{A}_r)$ for ζ and z such that $\Upsilon_{\zeta-a,z-a} \hookrightarrow \mathbf{K}$, while for γ_2 the series

$$(\zeta - z)^{-1} = -(z-a)^{-1}(\sum_{k=0}^{\infty}((\zeta-a)(z-a)^{-1})^k)$$

converges uniformly by $\zeta \in B(a, \rho_1, \rho_2-\epsilon, \mathcal{A}_r)$ and $z \in B(a, \rho_2, R_2, \mathcal{A}_r)$ for ζ and z such that $\Upsilon_{\zeta-a,z-a} \hookrightarrow \mathbf{K}$, for each $\rho' < \rho_2 < R_2 < R'$ and each $0 < \epsilon < \min(\rho_2-\rho_1, R_1-R_2)$, since expansion coefficients by $((\zeta-a)^{-1}(z-a))$ in the first series and by $((\zeta-a)(z-a)^{-1})$ in the second series are independent from the type of embedding. Consider corresponding γ_1 and γ_2 such that $\zeta \in \gamma_1$ or $\zeta \in \gamma_2$ respectively and a, ζ, z are subjected to the condition $\Upsilon_{\zeta-a,z-a} \hookrightarrow \mathbf{K}$, which is not finally restrictive due to Theorem 2.15. Consequently,

$$(3.23) \quad f(z)M = \sum_{k=0}^{\infty}(\phi_k(z) + \psi_k(z)), \text{ where}$$

$$\phi_k(z) := (2\pi)^{-1}\{\int_{\gamma_1} f(\zeta)((((\zeta-a)^{-1}(z-a))^k(\zeta-a)^{-1})d\zeta)\},$$

$$\psi_k(z) := (2\pi)^{-1}\{\int_{\gamma_2} f(\zeta)((z-a)^{-1}((\zeta-a)(z-a)^{-1})^k)d\zeta)\},$$

and where $\phi_k(z)$ and $\psi_k(z)$ are $\mathcal{A}_r$-holomorphic functions, hence fM has decomposition (3.21) in U, since due to §2.15 and §3.9 there exists $\delta > 0$ such that integrals for ϕ_k and ψ_k by γ_1 and γ_2 are the same for each $\rho' \in (\rho_1, \rho_1+\delta)$, $R' \in (R_1-\delta, R_1)$. Using the definition of the $\mathcal{A}_r$ line integral we get (3.21) converging on U. Varying $z \in U$ by $|z|$ and $Arg(z)$ we get that (3.21) converges absolutely on U, consequently, (3.20) is satisfied for fM. Since $M \in \mathcal{A}_r$, $|M| = 1$, is arbitrary, then as in the proof of Theorem 3.10 we get the statement of this theorem for f.

3.22. Notes and Definitions. Let γ be a closed curve in $\mathcal{A}_r$. There are natural projections from $\mathcal{A}_r$ on complex planes: $\pi_s(z) = w_1 + w_s s$ for each $s \in \hat{b}_r$, where $2 \le r \le \infty$, $z = \sum_{s \in \mathbf{b}_r} w_s s$ with real w_s for each $s \in \mathbf{b}_r$. Therefore, $\pi_s(\gamma) =: \gamma_s$ are curves in complex planes $\mathbf{C_s}$ isomorphic to $\mathbf{R} \oplus \mathbf{R}s$. A curve γ in $\mathcal{A}_r$ is closed (a loop, in another words) if and only if γ_s is closed for each $s \in \hat{b}_r$, that is, $\gamma(0) = \gamma(1)$ and $\gamma_s(0) = \gamma_s(1)$ correspondingly. In each complex plane there is the standard complex notion of a topological index $In(a_s, \gamma_s)$ of a curve γ_s at $a_s = \pi_s(a)$. Therefore, there exists a vector

$In(a,\gamma) := \{In(a_s,\gamma_s) : s \in \hat{b}_r\}$ which we call the topological index of γ at a point $a \in \mathcal{A}_r$. This topological index is invariant relative to homotopies satisfying conditions of Theorem 3.9. Consider now a standard closed curve $\gamma(s) = a + \rho \exp(2\pi t n M)$, where $M \in \mathcal{I}_r$ with $|M| = 1$, $n \in \mathbf{Z}$, $\rho > 0$, $t \in [0,1]$. Then $\hat{I}n(a,\gamma) := (2\pi)^{-1}(\int_\gamma dLn(z-a)) = nM$ is called the $\mathcal{A}_r$-index of γ at a point a. It is also invariant relative to homotopies satisfying the conditions of Theorem 3.9. Moreover, $\hat{I}n(h_1(ah_2), h_1(\gamma h_2)) = \hat{I}n(a,\gamma)$ for each h_1 and $h_2 \in \mathcal{A}_r \setminus \{0\}$ such that $h_1(Mh_2) = M$. For $M = \sum_{s\in\hat{b}_r} m_s s$ there is the equality $\hat{I}n(a,\gamma) = \sum_{s\in\hat{b}_r} In(a_s,\gamma_s) m_s s$ (adopting the corresponding convention for signs of indexes in each $\mathbf{C_s}$ and the convention of positive directions of going along curves). In view of the properties of Ln for each curve ψ in $\mathcal{A}_r$ there exists $\int_\gamma dLn(z-a) = 2\pi q M$ for some $q \in \mathbf{R}$ and $M \in \mathcal{I}_r$ with $|M| = 1$. For a closed curve ψ up to a composition of homotopies each of which is charaterized by homotopies in $\mathbf{C_s}$ for $s \in \hat{b}_r$ there exists a standard γ with a generator M for which $\hat{I}n(a,\gamma) = qM$, where $q \in \mathbf{Z}$. Therefore, we can take as a definiton $\hat{I}n(a,\psi) = \hat{I}n(a,\gamma)$. Define also the residue of a meromorphic function with an isolated singularity at a point $a \in \mathcal{A}_r$ as
(i) $res(a,f)M := (2\pi)^{-1}(\int_\gamma f(z)dz)$, where $\gamma(t) = a + \rho\exp(2\pi t M) \subset V$, $\rho > 0$, $|M| = 1$, $M \in \mathcal{I}_r$, $t \in [0,1]$, f is $\mathcal{A}_r$-holomorphic on $V \setminus \{a\}$. Extend $res(a,f)M$ by Formula (i) on $\mathcal{I}_r$ as
(ii) $res(a,f)M := [res(a,f)(M/|M|)]|M|$, $\forall M \ne 0$; $res(a,f)0 := 0$,
when $res(a,f)M$ is finite for each $M \in \mathcal{I}_r$, $|M| = 1$. For $r = 2$ or $r = 3$ the equation for $res(a,f)$ can be resolved for each $M \ne 0$.

If f has an isolated singularity at $a \in \hat{\mathcal{A}}_r$, then coefficients b_k of its Laurent series (see §3.21) are independent of $\rho > 0$. The common series is called the a-Laurent series. If $a = \infty$, then $g(z) := f(z^{-1})$ has a 0-Laurent series c_k such that $c_{-k} = b_k$. Let $\beta := \sup_{b_k \ne 0} \eta(k)$, where $\eta(k) = k_1 + ... + k_m$, $m = m(k)$ for $a = \infty$; $\beta = \inf_{b_k \ne 0} \eta(k)$ for $a \ne \infty$. We say that f has a removable singularity, pole, essential singularity at ∞ according as $\beta \le 0$, $0 < \beta < \infty$, $\beta = +\infty$. In the second case β is called the order of the pole at ∞. For a finite a the corresponding cases are: $\beta \ge 0$, $-\infty < \beta < 0$, $\beta = -\infty$. If f has a pole at a, then $|\beta|$ is called the order of the pole at a.

A value of a function $\partial_f(a) := \inf\{\eta(k) : b_k \ne 0\}$ is called a divisor of f at $a \ne \infty$, $\partial_f(a) := \inf\{-\eta(k) : b_k \ne 0\}$ for $a = \infty$, where $b_k \ne 0$ means that $b_{k,1} \ne 0$,....,$b_{k,m(k)} \ne 0$. Then $\partial_{f+g}(a) \ge \min\{\partial_f(a), \partial_g(a)\}$ for each $a \in dom(f) \cap dom(g)$ and $\partial_{fg}(a) = \partial_f(a) + \partial_g(a)$. For a function f

meromorphic on an open subset U in $\hat{\mathcal{A}}_r$ the function $\partial_f(p)$ by the variable $p \in U$ is called the divisor of f.

3.23. Theorem. *Let U be an open region in $\hat{\mathcal{A}}_r$, $2 \le r \le \infty$, with n distinct marked points $p_1, ..., p_n$, and let f be an $\mathcal{A}_r$-holomorphic function on $U \setminus \{p_1, ..., p_n\} =: U_0$ and ψ be a rectifiable closed curve lying in U_0 such that U_0 satisfies the conditions of Theorem 3.9 for each $z_0 \in \{p_1, ..., p_n\}$. Then*

$$\int_\psi f(z)dz = 2\pi \sum_{j=1}^n res(p_j, f)\hat{I}n(p_j, \psi)$$

and $res(p_j, f)M$ is the $\mathbf{R}$-homogeneous $\mathcal{I}_r$-additive (of the variable M in $\mathcal{I}_r$) $\mathcal{A}_r$-valued functional for each j.

Proof. For each p_j consider the principal part T_j of a Laurent series for f in a neighbourhood of p_j, that is,
$T_j(z) = \sum_{k,\eta(k)<0} \{(b_k, (z - p_j)^k)\}_{q(m(k)+\eta(k))}$, where $\eta(k) = k_1 + ... + k_n$ for $k = (k_1, ..., k_n)$ (see Theorem 3.21). Therefore, $h(z) := f(z) - \sum_j T_j(z)$ is a function having an $\mathcal{A}_r$-holomorphic extension on U. In view of Theorem 3.9 for an $\mathcal{A}_r$-holomorphic function g in a neighbourhood V of a point p and a rectifiable closed curve ζ we have

$$g(p)\hat{I}n(p, \zeta) = (2\pi)^{-1}(\int_\zeta g(z)(z - p)^{-1}dz)$$

(see §3.22). We may consider small ζ_j around each p_j with $\hat{I}n(p_j, \zeta_j) = \hat{I}n(p_j, \gamma)$ for each $j = 1, ..., n$. Then $\int_{\zeta_j} f(z)dz = \int_{\zeta_j} T_j(z)dz$ for each j. Representing U_0 as a finite union of open regions U_j and joining ζ_j with γ by paths ω_j going in one and the opposite direction as in Theorem 3.9 we get

$$\int_\gamma f(z)dz + \sum_j \int_{\zeta_j^-} f(z)dz = 0,$$

consequently,

$$\int_\gamma f(z)dz = \sum_j \int_{\zeta_j} f(z)dz = \sum_j 2\pi res(p_j, f)\hat{I}n(p_j, \gamma),$$

where $\hat{I}n(p_j, \gamma)$ and $res(p_j, f)$ are invariant relative to homotopies satisfying conditions of Theorem 3.9. Since $\int_{\zeta_j} g(z)dLn(z - p_j)$ is $\mathbf{R}$-homogeneous and $\mathcal{I}_r$-additive relative to a directing vector $M \in \mathcal{I}_r$ of a loop ζ_j, then

$res(p_j, f)M$ defined by Formulas 2.22.(i, ii) is $\mathbf{R}$-homogeneous $\mathcal{I}_r$-additive of the argument M in $\mathcal{I}_r$.

3.24. Corollary. *Let f and T be the same as in §3.23, then* $res(p_j, f)M = res(p_j, T_j)M = res(p_j, \sum_{k,\eta(k)=-1}\{(b_k, (z-p_j)^k)\}_{q(m(k)+\eta(k))}M$, *in particular,* $res(p_j, \{b(z-p_j)^{-1}c\}_{q(3)})M = \{bMc\}_{q(3)}$ *for each* $b, c \in \mathcal{A}_r$.

Proof. The first statement follows from §3.23, the second statement follows from left and right-$\mathcal{A}_r$-linearity of the line integral, though it is not the superlinear functional (see Theorem 2.7).

3.25. Corollary. *Let U be an open region in $\hat{\mathcal{A}}_r$, $2 \le r \le \infty$, with n distinct points $p_1, ..., p_n$, let also f be an $\mathcal{A}_r$-holomorphic function on $U \setminus \{p_1, ..., p_n\} =: U_0$, $p_n = \infty$, and U_0 satisfies conditions of Theorem 3.9 with at least one ψ, γ and each $z_0 \in \{p_1, ..., p_n\}$. Then $\sum_{p_j \in U} res(p_j, f)M = 0$.*

Proof. If γ is a closed curve encompassing p_1,...,p_{n-1}, then $\gamma^-(t) := \gamma(1-t)$, where $t \in [0, 1]$, encompasses $p_n = \infty$ with positive going by γ^- relative to p_n. Since $\int_\gamma f(z)dz + \int_{\gamma^-} f(z)dz = 0$, we get from Theorem 3.23, that $\sum_{p_j \in U} res(p_j, f)M = 0$ for each $M \in \mathcal{I}_r$, hence $\sum_{p_j \in U} res(p_j, f)M = 0$ is the zero $\mathbf{R}$-homogeneous $\mathcal{I}_r$-additive $\mathcal{A}_r$-valued functional on $\mathcal{I}_r$.

3.26. Definitions. Let f be an $\mathcal{A}_r$-holomorphic function, $2 \le r \le \infty$, on a neighbourhood V of a point $z \in \mathcal{A}_r$. Then the infimum: $\eta(z; f) := \inf\{k : k \in \mathbf{N}, f^{(k)}(z) \ne 0\}$ is called a multiplicity of f at z. Let f be an $\mathcal{A}_r$-holomorphic function on an open subset U in $\hat{\mathcal{A}}_r$, $2 \le r \le \infty$. Suppose $w \in \hat{\mathcal{A}}_r$, then the valence $\nu_f(w)$ of f at w is by the definition $\nu_f(w) := \infty$, when the set $\{z : f(z) = w\}$ is infinite, and otherwise $\nu_f(w) := \sum_{z, f(z)=w} \eta(z; f)$.

3.26.1. Theorem. *Let f be an $\mathcal{A}_r$-meromorphic right superlinearly superdifferentiable function on a region $U \subset \hat{\mathcal{A}}_r$. If $b \in \hat{\mathcal{A}}_r$ and $\nu_f(b) < \infty$, then b is not a cluster value of f and the set $\{z : \nu_f(z) = \nu_f(b)\}$ is a neighbourhood of b. If $U \ne \hat{\mathcal{A}}_r$ or f is not constant, then the converse statement holds. Nevertheless, it is false, when $f = const$ on $\hat{\mathcal{A}}_r$.*

3.26.2. Theorem. *Let U be a proper open subset of $\hat{\mathcal{A}}_r$, $2 \le r \le \infty$, let also f and g be two continuous functions from $\bar{U} := cl(U)$ into $\hat{\mathcal{A}}_r$ such that on a topological boundary $Fr(U)$ of U they satisfy the inequality $|f(z)| < |g(z)|$ for each $z \in Fr(U)$. Suppose f and g are $\mathcal{A}_r$-meromorphic functions in U and h be a unique continuous map from $\bar{U}$ into $\hat{\mathcal{A}}_r$ such that $h|_E = f|_E + g|_E$, where $E := \{z : f(z) \ne \infty, g(z) \ne \infty\}$, $[\partial Ln(h(z))/\partial z]$ is right superlinear in U_{z_0} for each zero z_0 and in $U_{z_0} \setminus \{z_0\}$ for each pole z_0, where U_{z_0} is a neighbourhood of z_0, $z \in U_{z_0}$ or $z \in U_{z_0} \setminus \{z_0\}$ respectively.*

Then $\nu_{g|_U}(0) - \nu_{g|_U}(\infty) = \nu_{h|_U}(0) - \nu_{h|_U}(\infty)$.

Proofs of these two theorems are analogous to that of Theorems VI.4.1, 4.2 [13]. To prove Theorem 3.26.2 consider the function $\zeta(z,t) := tf(z)+g(z)$ for each $z \in \bar{U}$ and each $t \in [0,1] \subset \mathbf{R}$. If z_0 is a pole of $h(z)$, then z_0 is a zero of $1/h(z)$. By the supposition of Theorem 3.26.1, Proposition 2.3 and Theorem 3.9 $res(z_0, Ln(h))$ is the right superlinear operator for each zero or pole z_0. On the other hand there exists $\delta > 0$ such that $\Delta_\gamma Arg(1+tg^{-1}f) = 0$ for each $t \in [-1,1]$, when no any pole or zero of g or f belongs to a rectifiable loop γ in U with $dist(\gamma, Fr(U)) < \delta$, where $dist(A,B) := \sup_{z\in A}(\inf_{\xi\in B}|\xi - z|) + (\sup_{\xi\in B}\inf_{z\in A}|\xi - z|)$. Then $\int_{z\in\gamma} dLn\zeta(z,t)$ is the continuous function by $t \in [0,1]$ taking values $2\pi nM$, where $M \in \mathcal{I}_r$ charterizes a rectifiable loop γ contained in U, $|M| = 1$, $n \in \mathbf{Z}$, M is independent of t. For each $\delta > 0$ it is possible to choose a rectifiable loop γ in U such that $dist(\gamma, Fr(U)) < \delta$. Then apply Theorem 3.23 to suitable pieces of U whose boundaries do not contain zeros and poles of f and g.

From the proof of Theorem 3.26.2 we get.

3.26.3. Corollary. *Let suppositions of Theorem 3.26 be satisfied may be besides the condition of right supelinearity of* $[\partial Ln(h(z))/\partial z]$*, then* $\Delta_{\partial U} Arg(f) = \Delta_{\partial U} Arg(g) = \int_\gamma dLn(f(z))$*, where* γ *is as in §3.26.2.*

3.27. Theorem. *Let* U *be an open subset in* $\mathcal{A}_r^n$*,* $2 \le r \le \infty$*, then there exists a represenation of the* $\mathbf{R}$*-linear space* $C^\omega_{z,\tilde{z}}(U, \mathcal{A}_r)$ *of locally* $(z,\tilde{z})$*-analytic functions on* U *such that it is isomorphic to the* $\mathbf{R}$*-linear space* $C^\omega_z(U, \mathcal{A}_r)$ *of* $\mathcal{A}_r$*-holomorphic functions on* U*.*

Proof. Evidently, the proof can be reduced to the case $n = 1$ by induction considering local $(z,\tilde{z})$-series decompositions by $({}^n z, {}^n\tilde{z})$ with coefficients being convergent series of $({}^1 z, {}^1\tilde{z}, ..., {}^{n-1}z, {}^{n-1}\tilde{z})$. We have
$\tilde{z} = (2^r-2)^{-1}\{-z + \sum_{s\in\hat{b}_r} s(zs^*)\}$ for each $2 \le r < \infty$,
$\tilde{z} = \lim_{r\to\infty}(2^r-2)^{-1}\{-z + \sum_{s\in\hat{b}_r} s(zs^*)\}$ in $\mathcal{A}_\infty$.
Consequently, each polynomial in $(z,\tilde{z})$ is also a polynomial in z only, moreover, each polynomial locally $(z,\tilde{z})$ analytic function on U is polynomial locally z-analytic on U. Then if a series by $(z,\tilde{z})$ converges in a ball $B(z_0, \rho, \mathcal{A}_r^n)$, then its series in the z-representation converges in a ball $B(z_0, \rho', \mathcal{A}_r^n)$, where $\rho' = 2^r(2^r-2)^{-1}\rho$. Considering basic polynomials of any polynomial basis in $C^\omega_{z,\tilde{z}}(U, \mathcal{A}_r)$ we get (due to infinite dimensionality of this space) a polynomial base of $C^\omega_z(U, \mathcal{A}_r)$. This establishes the $\mathbf{R}$-linear isomorphism between these two spaces. Moreover, in such representation of the space $C^\omega_{z,\tilde{z}}(U, \mathcal{A}_r)$ we can put $D_{\tilde{z}} = 0$, yielding for differential forms $\partial_{\tilde{z}} = 0$, this leads to differential

calculus and integration with respect to D_z and dz only.

3.28. Theorem (Argument principle). *Let f be an $\mathcal{A}_r$-holomorphic function on an open region U satisfying conditions of §3.9, $2 \le r \le \infty$, and let γ be a closed curve contained in U, where $[\partial Ln(f(z))/\partial z]$ is right supelinear in some neighbourhood U_{z_0} for each zero z_0 of $f(z)$. Then $\hat{I}n(0; f \circ \gamma) = \sum_{\partial_f(a)\ne 0} \hat{I}n(a;\gamma)\partial_f(a)$.*

Proof. There is the equality $\hat{I}n(0; f\circ\gamma) = \int_{\zeta\in\gamma} dLn(f(\zeta)) = \int_0^1 dLn(f \circ \gamma(s)) = \int_\gamma f^{-1}(\zeta)df(\zeta)$. Let $\partial_f(a) = n \in \mathbf{N}$, then

$$f^{-1}(a)f'(a).s = \sum_{l,k;n_1+...+n_k=\partial_f(a),0\le n_j\in\mathbf{Z},j=1,...,k} \{(z-a)^{n_1} g_{s,l,k,1;n_1,...,n_k}(z)$$

$$(z-a)^{n_2} g_{s,l,k,2;n_1,...,n_k}(z)...(z-a)^{n_k} g_{s,l,k,k;n_1,...,n_k}(z)\}_{q(n_1+...+n_k+k)},$$

where $g_{s,l,p,k;n_1,...,n_k}(z)$ are $\mathcal{A}_r$-holomorphic functions of z on U for each $s \in \mathbf{b}_r$ such that $g_{s,l,p,k;n_1,...,n_k}(a) \ne 0$, where $l = 1,...,m$, $1 \le m \le 2^{r\partial_f(a)}$ for finite r and each $m \in \mathbf{N}$ for $r = \infty$ (see §§2.8, 3.7, 3.21, 3.27), since each term $\xi(z)\prod_{s\in\mathbf{b}_r}(w_s - w_{s,0})^{n_s}$ with $\sum_{s\in\mathbf{b}_r} n_s \ge \partial_f(a)$, $n_j \ge 0$, has such decomposition, where $2 \le r < \infty$, $\xi(z)$ is an $\mathcal{A}_r$-holomorphic function on a neighbourhood of a such that $\xi(a) \ne 0$. When $r = \infty$ use $z = \lim_{r\to\infty} z_r$. Suppose ψ is a closed curve such that $\hat{I}n(p,\psi) = 2\pi n M$, $|M| = 1$, $M \in \mathcal{I}_r$, $0 \ne n \in \mathbf{Z}$. Then we can define a curve $\psi^{1/n} =: \omega$ as a closed curve for which $\hat{I}n(p,\omega) = 2\pi M$ and $\omega([0,1]) \subset \psi([0,1])$. Then we call $\omega^n = \psi$. That is, $\hat{I}n(p,\psi^{1/n}) = \hat{I}(p,\psi)/n$. The latter formula allows an interpretation also when $\hat{I}n(p,\psi)/n$ is equal to $2\pi qM$, where $0 \ne q \in \mathbf{Q}$. That is, a curve $\psi^{1/n}$ can be defined for each $0 \ne n \in \mathbf{Z}$. This means that γ can be presented as union of curves ω_j for each of which there exists $n_j \in \mathbf{N}$ such that $\omega_j^{n_j}$ is a closed curve. Using Theorem 3.9 for each $a \in U$ with $\partial_f(a) \ne 0$, also using the series given above we can find a finite family of ω_j for which one of the terms in the series is not less, than any other term. We may also use small homotopic deformations of ω_j satisfying the conditions of Theorem 3.9 such that in the series one of the terms is greater than any other for almost all points on ω_j. Such deformation is permitted, since otherwise two terms would coincide on an open subset of U, that is impossible. Considering such series, Formulas 2.5.(4, 5) and using Theorem 3.26.2 we get the statement of this theorem.

3.29. Theorem. *If f has an essential singularity at a, then $cl(f(V)) = \hat{\mathcal{A}}_r$ for each $V \subset dom(f)$, $V = U \setminus \{a\}$, where U is a neighbourhood of a.*

Proof. Suppose that the statement of this theorem is false, then there would exist $\rho > 0$ and $m > 0$ and an element $A \in \mathcal{A}_r$ such that f is z-analytic in $B(a, 0, \rho, \mathcal{A}_r) \setminus \{a\}$ and $|f(z) - A| \ge m$ for each z such that $0 < |z-a| < \rho$. If $\infty \notin cl(f(V))$, then there exists $R > 0$ such that $A \notin cl(f(V))$ for each $|A| > R$. Therefore, the function $[f(z) - A]^{-1}$ is $\mathcal{A}_r$-holomorphic in $B(a, 0, \rho, \mathcal{A}_r) \setminus \{a\}$. Hence $[f(z)-A]^{-1} = \sum_k \{(p_k, (z-a)^k)\}_{q(\eta(k)+m(k))}$, where in this sum $k = (k_1, ..., k_{m(k)})$ with $k_j \ge 0$ for each $j = 1, ..., m(k) \in \mathbf{N}$, p_k are finite sequences of coefficients for $[f(z) - A]^{-1}$ as in §3.21. If $D^n_z([f(z) - A]^{-1})|_{z=a} = 0$ for each $n \ge 0$, then $[f(z) - A]^{-1} = 0$ in a neighbourhood of a. Therefore, $[f(z) - A]^{-1} = \sum_{n_1+...+n_l=n} \{g_1 z^{n_1} ... g_l z^{n_l}\}_{q(\eta(n)+l)}$ for some n such that $0 \le n \in \mathbf{N}$, $n_j \ge 0$ for each $j = 1, ..., l \in \mathbf{N}$, each g_j is an $\mathcal{A}_r$-holomorphic function (of z). Consequently, taking inverses of both sides $[f(z) - A]$ and $(\sum_{n_1+...+n_l=n} \{g_1 z^{n_1} ... g_l z^{n_l}\}_{q(\eta(n)+l)})^{-1}$ and comparing their expansion series we see that finite sequences b_k of expansion coefficients for f have the property $b_k = 0$ for each $\eta(k) < -n$. This contradicts the hypothesis and proves the theorem.

3.30. Definition. Let a and b be two points in $\mathcal{A}_r$ and θ be a stereographic mapping of the unit real sphere S^{2^r} for $2 \le r < \infty$ or S^∞ for $r = \infty$ on $\hat{\mathcal{A}}_r$. Then $\chi(a, b) := |\phi(a) - \phi(b)|_Y$ is called the chordal metric, where $\phi := \theta^{-1} : \hat{\mathcal{A}}_r \to S^m$, S^m is embedded in $Y := \mathbf{R^{m+1}}$ for $m := 2^r$ with $r < \infty$ or in $\mathbf{R} \oplus l_2(\mathbf{R})$ for $m = \infty$ with $r = \infty$, $| * |_Y$ is the Euclidean or Hilbert norm in Y respectively.

3.30.1. Theorem. *Let U be an open region in $\hat{\mathcal{A}}_r$, $\{f_n : n \in \mathbf{N}\}$ be a sequence of functions meromorphic on U tending uniformly in U to f relative to the chordal metric. Then either f is the constant ∞ or else f is meromorphic on U.*

3.30.2. Theorem. *Let $\{f_k : k \in \mathbf{N}\}$ be a sequence of meromorphic functions on an open subset U in $\hat{\mathcal{A}}_r$, which tends uniformly in the sence of the chordal metric in U to f, $f \ne const$. If $f(a) = b$ and $\rho > 0$ are such that $B(a, \rho, \mathcal{A}_r) \subset U$ and $f(z) \ne b$ for each $z \in B(a, \rho, \mathcal{A}_r) \setminus \{a\}$, then there exists $m \in \mathbf{N}$ such that the value of the valence of $f_k|_{B(a,\rho,\mathcal{A}_r)}$ at b is $\eta(b; f) = \eta(a; f)$ for each $k \ge m$.*

3.30.3. Note. The proofs of these theorems are formally similar to the proofs of VI.4.3 and 4.4 [13]. Theorems 3.26.2 and 3.30.2 are the $\mathcal{A}_r$ analogs of the Rouché and Hurwitz theorems respectively. There are also the following $\mathcal{A}_r$ analogs of the Mittag-Leffler and Weierstrass theorems. Their proofs are similar to those for Theorems VIII.1.1 and 1.2 [13] respectively. Nevertheless

the second part of the Weierstrass theorem is not true in general because of noncommutativity of $\mathcal{A}_r$, that is, a function $h \in \mathbf{M}(U)$ with $\partial_h = \partial$ is not necessarily representable as $h = fg$, where g is an $\mathcal{A}_r$-holomorphic on U and f is another marked function $f \in \mathbf{M}(U)$ such that $\partial_f = \partial$.

3.31. Theorem. *Let U be a nonempty proper open subset of $\hat{\mathcal{A}}_r$, $2 \le r \le \infty$, let $A \subset U$ not containing any cluster point in U. Let there be a function $g_b \in \mathbf{M}(\hat{\mathcal{A}}_r)$ for each $b \in A$ having a pole at b and no other. Then there exists $f \in \mathbf{M}(U)$ $\mathcal{A}_r$-holomorphic on $U \setminus B$ and having the same principal part at b as g_b. If f is such a function, then each other such function is the function $f + g$, where g is $\mathcal{A}_r$-holomorphic on U.*

3.32. Theorem. *Let U be a proper nonempty open subset of $\hat{\mathcal{A}}_r$, $2 \le r \le \infty$. Let $\partial : U \to \mathbf{Z}$ be a function such that $\{\partial(z) \ne 0\}$ does not have a cluster point in U. Then there exists $f \in \mathbf{M}(U)$ such that $\partial_f = \partial$.*

Proof. If a_j is a zero, that is, $\partial(a_j) \ne 0$, then take a circle of radius $\delta_j > 0$ with centre at a_j. There are possible two cases: $|a_j|\delta_j \ge 1$ and $|a_j|\delta_j < 1$. At a_j of the first type construct in U a meromorphic function $g(z)$ with the principal part $g_j(z) = n_j(z - a_j)^{-1}$, $n_j = \partial(a_j)$, $g(z) := \sum_{j=1}^{\infty}(g_j(z) - h_j(z))$, where $h_j(z) := -n_j \sum_{p=1}^{k_j}(a_j^{-1}z)^{p-1}a_j^{-1}$. Choose k_j such that in each bounded canonical closed V in $\mathcal{A}_r$, $V \subset U$, the series for g is unifomly convergent. At a_j of the second type construct
$g_j(z) := n_j \sum_{p=0}^{\infty}(z - b_j)^{-1}[(a_j - b_j)(z - b_j)^{-1}]^p$ and
$h_j(z) := n_j \sum_{p=0}^{k_j}(z - b_j)^{-1}[(a_j - b_j)(z - b_j)^{-1}]^p$,
where $|a_j - b_j| < |z - b_j|$, a_j, b_j, z are subjected to the condition $\Upsilon_{a_j - b_j, z - b_j} \hookrightarrow \mathbf{K}$, which is not finally restrictive due to Theorem 2.15. Choose k_j such that for each z with $|z - b_j| \ge R > \delta_j$ we have $|g_j(z) - h_j(z)| = |n_j \sum_{p=k_j+1}^{\infty}(z - b_j)^{-1}[(a_j - b_j)(z - b_j)^{-1}]^p| \le n_j \sum_{p=k_j+1}^{\infty} |(a_j - b_j)^p R_j^{-p-1}| < \epsilon_j$, where $\sum_{j=1}^{\infty} \epsilon_j < \infty$ converges, $R_j > 0$ is a constant for each j. These series by $(a_j^{-1}z)^p a_j^{-1}$ or by $(z - b_j)^{-1}[(a_j - b_j)(z - b_j)^{-1}]^p$, $p = 0, 1, 2, ...$, are with real coefficients independent of the type of embedding of $\mathbf{K}$ into $\mathcal{A}_r$, where $z \in \Upsilon_{a_j, b_j, z_1} \hookrightarrow \mathbf{K} \hookrightarrow \mathcal{A}_r$ for each given $z_1 \in \mathcal{A}_r$ with the variable z within a given copy of $\mathbf{K}$, hence they can be extended on the corresponding balls in $\mathcal{A}_r$. Now integrate $g(\zeta)$ along a rectifiable path γ in U which does not contain any a_j, $\gamma(0) = z_0$, $\gamma(1) = z$. Then $\int_{z_0}^{z} g(\zeta)d\zeta = \sum_{j=1}^{\infty}[w_j(z) - w_j(z_0)]$ such that $f_1(z) := \exp(\int_{z_0}^{z} g(\zeta)d\zeta)$ is independent of the path (see Theorems 2.15 and 3.8.3 and Corollary 3.4 above), where
$\exp(w_j(z)) = ((1 - a_j^{-1}z)\exp(\sum_{p=1}^{\infty}(a_j^{-1}z)^p/p]))^{n_j}$ in the first case,

$\exp(w_j(z)) = ([(z-a_j)(z-b_j)^{-1}]\exp(\sum_{p=1}^\infty[(a_j-b_j)(z-b_j)^{-1}]^p/p))^{n_j}$ in the second case, but w_j and w_l generally do not commute for $j \ne l$. The convergence of series is analogous to that of Satz 25 [2] in the complex case. Two functions satisfying Theorem 3.32 need not differ on an $\mathcal{A}_r$-holomorphic multiplier apart from the complex case, since, for example, $\zeta_j := \{f_1...f_j g f_{j+1}...f_n\}_{q(n)}$ with different $j = a$ and $j = b$ in $\{1,...,n\} \subset \mathbf{N}$ do not correlate: $\zeta_a \ne \{h\zeta_b k\}_{q(3)}$ in general for any $\mathcal{A}_r$-holomorphic h and k functions on U, where each f_l has a zero of order $n_l > 0$ at a_l, $n \ge 2$. Moreover, for $r \ge 3$ there is also dependence on the order of multiplication $\{*\}_{q(n)}$.

3.33. Theorem. *Let U be an open region in $\mathcal{A}_r$, $2 \le r \le \infty$, and f be a function $\mathcal{A}_r$-holomorphic on U with a right superlinear superdifferential on U. Suppose f is not constant and $B(a,\rho,\mathcal{A}_r) \subset U$, where $0 < \rho < \infty$. Then $f(B(a,\rho,\mathcal{A}_r))$ is a neighbourhood of $f(a)$ in $\mathcal{A}_r$.*

3.34. Remarks. For several $\mathcal{A}_r$ variables a multiple $\mathcal{A}_r$ line integral $\mathbf{I} := \{\int_{\gamma_n}...\int_{\gamma_1} f({}^1z,...,{}^nz d\,{}^1z...d\,{}^nz\}_{q(n)}$ may be naturally considered for rectifiable curves γ_1,...,γ_n in $\mathcal{A}_r$, $3 \le r \le \infty$, where $\{*\}_{q(n)}$ denotes the order of brackets in the order or iterated integrations (see also §2.1). Generally, the order of integration is essential, since the existence of the partial derivative $\partial^n g({}^1z,...,{}^nz)/\partial\,{}^1z...\partial\,{}^nz$ does not imply an existence of a continuous $g^{(n)}$ and as Proposition 3.8.5 shows the order of differentiation is essential, for example, even in the case of g corresponding to $f := \{f_1 \circ ... \circ f_n\}_{q(n)}$ with $f_n({}^1z,...,{}^nz)$ with values in $\mathcal{A}_r^{n-1}$, $f_{n-1}({}^1z,...,{}^{n-1}z)$ with values in $\mathcal{A}_r^{n-2}$, ...,$f_2({}^1z,{}^2z)$ and $f_1({}^1z)$ with values in $\mathcal{A}_r$, $\{\partial^n g({}^1z,...,{}^nz)/\partial\,{}^1z...\partial\,{}^nz\}_{q(n)}.1^{\otimes n} = f({}^1z,...,{}^nz)$ (see also §2.7). Therefore, there is the natural generalization of Theorem 3.9 for several $\mathcal{A}_r$ variables:

$$(i)\quad (2\pi)^n f(z_0)\{M_1...M_n\}_{q(n)} =$$

$$\{\int_{\psi_n}...\int_{\psi_1} f({}^1\zeta,...,{}^n\zeta)({}^1\zeta - {}^1z_0)^{-1}d\,{}^1\zeta)...({}^n\zeta - {}^nz_0)^{-1}d\,{}^n\zeta)\}_{q(n)}$$

for the corresponding $U = {}^1U \times ... \times {}^nU$, where ψ_j and ${}^jU$ satisfy conditions of Theorem 3.9 for each j and f is a continuous $\mathcal{A}_r$-holomorphic function on U.

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Address: Sergey V. Ludkovsky, Mathematical Department,
Brussels University, V.U.B., Pleinlaan 2, Brussels 1050, Belgium.
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